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## Matching the $D^6\mathcal{R}^4$ interaction at two-loops

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**ABSTRACT:** The coefficient of the  $D^6\mathcal{R}^4$  interaction in the low energy expansion of the two-loop four-graviton amplitude in type II superstring theory is known to be proportional to the integral of the Zhang-Kawazumi (ZK) invariant over the moduli space of genus-two Riemann surfaces. We demonstrate that the ZK invariant is an eigenfunction with eigenvalue 5 of the Laplace-Beltrami operator in the interior of moduli space. Exploiting this result, we evaluate the integral of the ZK invariant explicitly, finding agreement with the value of the two-loop  $D^6\mathcal{R}^4$  interaction predicted on the basis of S-duality and supersymmetry. A review of the current understanding of the  $D^{2p}\mathcal{R}^4$  interactions in type II superstring theory compactified on a torus  $T^d$  with  $p \leq 3$  and  $d \leq 4$  is included.

**KEYWORDS:** Supersymmetry and Duality, String Duality

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Outline	5
<b>2</b>	<b>Constraints from S-duality and supersymmetry</b>	<b>5</b>
2.1	The full four-graviton amplitude	5
2.2	Perturbative contributions to the low energy expansion	6
2.2.1	Genus zero	8
2.2.2	Genus one	8
2.2.3	Genus two	9
2.2.4	Genus three	10
2.3	S-duality and differential constraints	11
2.3.1	S-duality constraints on the perturbative coefficients	12
<b>3</b>	<b>The Zhang-Kawazumi invariant</b>	<b>14</b>
3.1	Differential constraint on the ZK invariant: a first hint	15
3.2	The ZK invariant in the supergravity limit	15
<b>4</b>	<b>The Laplacian of the Zhang-Kawazumi invariant</b>	<b>17</b>
4.1	Preliminaries	18
4.2	Basic variational formulas	18
4.3	Calculation of the first variational derivative	19
4.4	Calculation of the second variational derivative	19
4.5	Calculation of mixed variations for genus two	21
4.6	Calculation of $\Delta\varphi$ for genus two	21
<b>5</b>	<b>Integrating the ZK invariant over moduli space</b>	<b>22</b>
5.1	Convergence and regularization near the separating node	23
5.2	Reducing the integral to the boundary of moduli space	24
5.3	Calculation of the integral $\int d\mu_2 \varphi$	25
5.4	Differential relation for the Faltings invariant	26
<b>A</b>	<b>Some modular geometry</b>	<b>27</b>
A.1	Metric and volume	27
A.2	The Laplace-Beltrami operator	28
A.3	Moduli spaces of low genus	29
<b>B</b>	<b>Calculation of the mixed variation of <math>\varphi</math></b>	<b>30</b>
<b>C</b>	<b>Calculation of the Laplacian of <math>\varphi</math> for genus <math>h \geq 3</math></b>	<b>31</b>
C.1	The contribution to $\Delta\varphi$ which vanishes for $h = 2$	32

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# 1 Introduction

Much of our current understanding of superstring theory is based on two different expansion schemes: the genus expansion, and the low-energy expansion. Our aim is to compute a particular term in this double expansion, namely the two-loop contribution to the  $D^6\mathcal{R}^4$  interaction in type II superstrings, and check agreement with predictions from dualities and supersymmetry.

The genus expansion, on the one hand, is an asymptotic series in the string coupling,  $g_s$ , valid at small coupling. A term of order  $g_s^{2h-2}$  is associated with an integral over the moduli space of genus- $h$  (super)Riemann surfaces, generalizing  $h$ -loop Feynman diagrams in quantum field theory. Obtaining explicit expressions for scattering amplitudes in string perturbation theory has been possible up to genus-two [1, 2] (see [3] for a recent overview), but higher genus contributions remain largely elusive. Non-perturbative contributions to scattering amplitudes are in general unknown.

The low-energy expansion, on the other hand, is valid for small momenta, in units of the inverse string length scale  $1/\sqrt{\alpha'}$ . The leading term in this expansion reproduces the tree-level amplitudes of supergravity, while terms of higher order in  $\alpha'$  describe local and non-local higher derivative effective interactions. The coefficient in front of each of these terms is a function of the moduli, including the string coupling  $g_s$  and the parameters describing the target space, and receives both perturbative and non-perturbative contributions. The first few local interactions in the low-energy expansion are typically determined at low order in string perturbation theory, up to non-perturbative contributions which are constrained, and sometimes uniquely fixed, by supersymmetry and duality.

We will concentrate on flat maximally supersymmetric backgrounds of type IIB superstring theory. For the simplest such background, namely ten-dimensional Minkowski space-time, the amplitudes are expected to be invariant under the action of the S-duality group  $SL(2, \mathbb{Z})$ . This group acts on the single complex modulus field  $T$  (the axion-dilaton) by Möbius transformations. We shall also consider partially compactified space-times of the form  $\mathbb{R}^{10-d} \times \mathbb{T}^d$ , where  $\mathbb{T}^d$  is a flat torus of dimension  $d$ . The moduli space in this case includes, in addition to the axion-dilaton  $T$ , the constant metric  $G$  and 2-form field  $B$  on  $\mathbb{T}^d$ , along with the Ramond-Ramond potentials (and when  $d \geq 6$ , Neveu-Schwarz axions). The corresponding set of all moduli, denoted by  $m_d$ , parametrizes the symmetric coset spaces<sup>1</sup>  $E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$ . The latter may be viewed as a fiber bundle over  $\mathbb{R}^+ \times SO(d, d)/SO(d) \times SO(d)$ , where the first factor corresponds to the  $(10 - d)$ -dimensional string coupling  $g_d = g_s/\sqrt{\det G}$  while the second factor parametrizes the moduli  $\rho_d = (G, B)$ . For  $d \geq 8$  the notion of moduli space becomes ill-defined.

Moreover, we restrict attention to higher derivative local interactions of the form  $D^{2p}\mathcal{R}^4$ , with  $0 \leq p \leq 3$ , where  $\mathcal{R}^4$  indicates the particular tensorial contraction of four Riemann tensors dictated by supersymmetry, and  $D^{2p}$  stands for a certain combination of

<sup>1</sup>The rank  $n$  Lie group  $E_n(\mathbb{R})$  coincides with the (non-compact) real split form  $E_{n(n)}$  of the exceptional Lie group  $E_n$  for  $n = 8, 7, 6$ , while for the ranks  $n = 5, 4, 3, 2, 1$ , the groups  $E_n(\mathbb{R})$  are respectively given by  $SO(5, 5, \mathbb{R})$ ,  $SL(5, \mathbb{R})$ ,  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ ,  $SL(2, \mathbb{R}) \times \mathbb{R}^+$ , and  $SL(2, \mathbb{R})$ . The group  $K_n(\mathbb{R})$  is the maximal compact subgroup of  $E_n(\mathbb{R})$ .

$2p$  covariant derivatives to be described below. Effective interactions with only two or three powers of the Riemann tensor are forbidden by supersymmetry in type II. The restriction to  $p \leq 3$  ensures that the interactions are related by supersymmetry to fermionic vertices with strictly fewer than 32 fermions, and hence will be protected from non-BPS contributions. We will use the notation  $\mathcal{E}_{(0,0)}(m_d) \mathcal{R}^4$ ,  $\mathcal{E}_{(1,0)}(m_d) D^4 \mathcal{R}^4$ , and  $\mathcal{E}_{(0,1)}(m_d) D^6 \mathcal{R}^4$  to denote these effective interactions<sup>2</sup> in the Einstein frame.

The coefficients  $\mathcal{E}_{(m,n)}(m_d)$  are functions on the symmetric space  $E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$ , invariant under the dualities of string theory. In particular, they are invariant under the T-duality group  $\text{SO}(d, d, \mathbb{Z})$ , which leaves the  $(10 - d)$ -dimensional string coupling  $g_d$  invariant. This symmetry therefore holds at each order in the expansion in powers of  $g_d$ . A far more powerful statement is that the coefficients  $\mathcal{E}_{(m,n)}(m_d)$  should also be invariant under an arithmetic subgroup  $E_{d+1}(\mathbb{Z})$  of  $E_{d+1}(\mathbb{R})$  known as U-duality [4, 5]. It arises by combining T-duality with the  $\text{SL}(d + 1, \mathbb{Z})$  symmetry manifest in the M-theory lift of type II string theory, and reduces to the S-duality group  $\text{SL}(2, \mathbb{Z})$  for  $d = 0$ . U-duality relates perturbative and non-perturbative contributions and, along with constraints arising from supersymmetry, often determines the exact form of these coefficient functions. For example, in ten-dimensional Minkowski space (namely  $d = 0$ ),  $\mathcal{E}_{(0,0)}(T)$  and  $\mathcal{E}_{(1,0)}(T)$ , are constrained by supersymmetry to be eigenmodes of the Laplacian on the upper half  $T$ -plane with eigenvalues  $3/4$  and  $15/4$ , respectively, and are therefore proportional to the non-holomorphic Eisenstein series  $E^*(\frac{3}{2}, T)$  and  $E^*(\frac{5}{2}, T)$  [6–10]. Similar results hold for  $d > 0$ , with  $\mathcal{E}_{(0,0)}(m_d)$  and  $\mathcal{E}_{(1,0)}(m_d)$  being combinations of Eisenstein series of the U-duality group [7, 11–15]. These Eisenstein series reproduce the known perturbative contributions up to one and two-loop, respectively, and for  $d = 2$  can be derived from one-loop and two-loop supergravity amplitudes in M-theory compactified on  $T^2$  [9, 16]. Like any Eisenstein series, (or residue thereof) they are quasi-eigenmodes of the Laplace-Beltrami operator on  $E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$ , see (2.29)–(2.30) below, although this property has not yet been derived from supersymmetry.

The situation for the coefficient  $\mathcal{E}_{(0,1)}$  of the  $D^6 \mathcal{R}^4$  interaction is more challenging, as it is clear from the analysis of 2-loop supergravity amplitudes that  $\mathcal{E}_{(0,1)}$  cannot be an eigenmode of the Laplacian, rather it must satisfy an inhomogeneous Laplace equation, eq. (2.31) below, with a source proportional to the square of  $\mathcal{E}_{(0,0)}$ , see [12, 13, 17]. The solution of this equation contains perturbative terms that correspond to zero-, one-, two-, and three-loop contributions in superstring theory. The tree-level and one-loop values were verified by perturbative string theory calculations for  $d = 0$  in [18], and further agreement has also been found for  $d \leq 4$  [12] and  $d = 7$  [13]. Furthermore, the predicted three-loop value agrees with the value of the coefficient of  $D^6 \mathcal{R}^4$  in the type IIA theory predicted by an S-duality argument in [9]. There is also a claimed agreement with an explicit three-loop string theory calculation in [19] (although a potential mismatch by a factor of 3 needs to be sorted out).

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<sup>2</sup>The linearised  $D^2 \mathcal{R}^4$  contribution to the four-graviton amplitude vanishes for kinematic reasons and the non-linear interaction is believed to vanish identically. The coefficients  $\mathcal{E}_{(m,n)}(m_d)$  multiply effective interactions schematically represented by  $D^{2p} \mathcal{R}^4$  with  $p = 2m + 3n$ , as will be explained in section 2.1.

The only remaining S-duality prediction still to be verified in superstring perturbation theory is the two-loop contribution to the coefficient of  $D^6\mathcal{R}^4$ , which will be denoted by  $\mathcal{E}_{(0,1)}^{(2)}$ . The predicted value for  $d = 0$  is,

$$\mathcal{E}_{(0,1)}^{(2)} = \frac{2\pi^4}{45} \quad (1.1)$$

This value should emerge from the low energy expansion of the two-loop four-graviton string amplitude obtained in [1, 20].<sup>3</sup> While the two-loop contributions to the  $\mathcal{R}^4$  and  $D^4\mathcal{R}^4$  interactions were relatively straightforward to extract from the explicit two-loop four-graviton amplitude [26], the two-loop contribution to  $D^6\mathcal{R}^4$  involves various mathematical subtleties. As a step towards its evaluation, the value of  $\mathcal{E}_{(0,1)}^{(2)}$  arising from an analysis of the two-loop amplitude was expressed in [27] as the integral,

$$\mathcal{E}_{(0,1)}^{(2)} = \pi \int_{\mathcal{M}_2} d\mu_2 \varphi \quad (1.2)$$

Here  $d\mu_2$  is the canonical volume form on  $\mathcal{M}_2$  and  $\varphi$  may be expressed as follows,

$$\varphi(\Sigma) = -\frac{1}{8} \int_{\Sigma^2} P(x, y) G(x, y) \quad (1.3)$$

where  $G(x, y)$  is the scalar Green function on the genus-two surface,  $\Sigma$ , and  $P(x, y)$  is a bi-holomorphic form in  $x, y \in \Sigma$  that is defined in a manner that makes (1.3) conformal invariant, as will be reviewed in section 3. The object  $\varphi$  in (1.3) is an invariant of the Riemann surface<sup>4</sup>  $\Sigma$  that was discovered by Zhang [28] and Kawazumi [29]. The equivalence of (1.3) with other definitions of the Zhang-Kawazumi (ZK) invariant in the mathematical literature was derived in [27]. In spite a variety of available reformulations, the direct integration of  $\varphi$  on the moduli space  $\mathcal{M}_2$  of genus-two Riemann surfaces remained elusive.

In this paper, we shall evaluate the integral in (1.2) and thereby find agreement between the value predicted by S-duality, (1.1), and the result of two-loop superstring perturbation theory. This matching is highly non-trivial, and involves some novel mathematics. The key observation which makes this computation possible is the fact that  $\varphi$  satisfies the following differential equation everywhere in the interior of  $\mathcal{M}_2$ ,

$$(\Delta - 5) \varphi = 0 \quad (1.4)$$

where  $\Delta$  is the Laplace-Beltrami operator on  $\mathcal{M}_2$  associated with the Poincaré metric on the Siegel upper half space  $\mathcal{S}_2$ , to be defined in appendix A below. Equation (1.4) will be derived here from first principles by using the theory of deformation of complex structures. The validity of (1.4) may be extended to the Deligne-Mumford compactification

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<sup>3</sup>The construction of the two-loop gauge invariant measure was obtained in [21, 22]; a construction of the measure based on holomorphy and modular invariance was given in [23]; gauge invariance of the two-loop four-graviton amplitude was proven in [24]; the amplitude was reproduced in the pure spinor formulation and extended to including external fermions in [25]

<sup>4</sup>The definition (1.3) holds for a Riemann surface of any genus  $h \geq 2$ , upon replacing 8 by  $4h$  in the denominator. The ZK invariant vanishes for  $h = 0, 1$ .

$\overline{\mathcal{M}}_2$  of moduli space, upon supplementing (1.4) by a term with  $\delta$ -function support on the separating node,

$$(\Delta - 5)\varphi = -2\pi\delta_{SN} \quad (1.5)$$

an equation which is valid throughout  $\overline{\mathcal{M}}_2$ . We note that the combination  $\partial\bar{\partial}\varphi$  was evaluated in [29, 30] for arbitrary genus. It would be interesting to understand its connection with (1.4) and (1.5), if any exists, especially since the ZK invariant for genus  $h \geq 3$  does not satisfy a simple equation of the type (1.4), as will be shown in appendix C.

## 1.1 Outline

The layout of this paper is as follows. In section 2 we give a brief overview of perturbative and non-perturbative features of the low energy expansion of the four-graviton amplitude, both in flat 10-dimensional space-time and upon compactification on  $d$ -dimensional flat tori with  $d \leq 4$ . The differential equations imposed by S-duality and supersymmetry on the coefficient functions,  $\mathcal{E}_{(m,n)}(m_d)$ , are reviewed, and their implications on the perturbative expansion of these coefficients are obtained. In section 3, the differential constraints mentioned above are recast in the form of constraints on certain integrals of the ZK invariant  $\varphi$ , which suggest a Laplace eigenvalue equation for  $\varphi$ . Further evidence is provided by the fact that the supergravity limit of  $\varphi$  satisfies this equation. In section 4, the Laplace eigenvalue equation for  $\varphi$  is proven for genus-two using the methods of deformations of complex structures on Riemann surfaces. In section 5 the Laplace eigenvalue equation is used to integrate  $\varphi$  over moduli space, and thereby prove the matching of the coefficient of the  $D^6\mathcal{R}^4$  interaction. We end by providing the corresponding differential equations for other genus-two modular forms including the Faltings invariant. Basic facts about modular geometry are collected in appendix A; details of the calculation of the Laplacian are given in appendix B while its generalization to higher genus is obtained in appendix C.

## 2 Constraints from S-duality and supersymmetry

In this section, we shall describe relevant aspects of the  $\alpha'$  expansion for the four-graviton amplitude in type IIB string theory compactified on  $\mathbb{R}^{10-d} \times \mathbb{T}^d$  for  $0 \leq d \leq 4$ , and of the differential equations satisfied by the coefficient functions  $\mathcal{E}_{(m,n)}$  of the various BPS effective interactions.

### 2.1 The full four-graviton amplitude

The full four-graviton amplitude  $\mathcal{A}^{(4)}(\epsilon_i, k_i; m_d)$  depends on the  $(10-d)$ -dimensional polarization tensors  $\epsilon_i$  and momenta  $k_i$  for  $i = 1, 2, 3, 4$ , as well as on the moduli  $m_d$ . The dependence on  $d$  will be understood throughout, but not exhibited. The maximal supersymmetry of type IIB string theory singles out a unique tensorial structure for the full four-graviton amplitude which we denote by  $\mathcal{R}^4$ , and which is a special contraction of four Riemann tensors. Its explicit form may be found in [27], for example, but will not be needed here. As a result, we may introduce a reduced four-graviton amplitude  $\mathcal{I}$ , defined by,

$$\mathcal{A}^{(4)}(\epsilon_i, k_i; m_d) = \kappa_d^2 \mathcal{R}^4 \mathcal{I}(s, t, u; m_d) \quad (2.1)$$

where  $\kappa_d^2$  is Newton's constant in  $10 - d$  dimensions. The reduced amplitude  $\mathcal{I}$  is a dimensionless function of the moduli  $m_d$  and the dimensionless variables  $s = -\alpha'(k_1 + k_2)^2/4$ ,  $t = -\alpha'(k_1 + k_4)^2/4$  and  $u = -\alpha'(k_1 + k_3)^2/4$ , subject to the relation  $s + t + u = 0$  which follows from momentum conservation and the mass-shell conditions  $k_i^2 = 0$ . Our interest is in the low energy expansion of  $\mathcal{I}$ , or  $\alpha'$  expansion, in which  $|s|, |t|, |u| \ll 1$ .

In this limit, the reduced amplitude  $\mathcal{I}(s, t, u; m_d)$  can be decomposed into a sum of terms that are analytic in  $s, t, u$ , corresponding to local interactions in the effective action, and terms that are non-analytic, corresponding to non-local effective interactions. Non-local terms arise from integrating out massless states, and are computable in the framework of perturbative supergravity, supplemented by appropriate counter-terms to remove ultraviolet divergences. The remainder is an analytic function of  $s, t, u$  which contains contributions from massive string states. The local part, of interest in this work, is related to, and constrained by, the non-local part via unitarity. In the Einstein frame, the local part has an expansion of the form,

$$\mathcal{I}(s, t, u; m_d) \Big|_{\text{local}} = \frac{3}{\sigma_3} + \sum_{m,n=0}^{\infty} \mathcal{E}_{(m,n)}(m_d) \sigma_2^m \sigma_3^n \quad (2.2)$$

The first term in (2.2) gives the tree-level amplitude of type II supergravity, while the higher-order terms give higher-derivative local interactions. Because of the relation  $s + t + u = 0$ , the expansion is in powers of only two independent variables, chosen to be  $\sigma_2 = s^2 + t^2 + u^2$  and  $\sigma_3 = s^3 + t^3 + u^3$  [18]. The variable  $g_d$  is the effective string coupling defined by  $g_d = g_s$  when  $d = 0$  and  $g_d = g_s/\sqrt{\mathcal{V}_d}$  for  $d > 0$  where  $\mathcal{V}_d$  is the volume of  $\mathbb{T}^d$  with metric  $G$ .

In the Einstein frame, the reduced amplitude  $\mathcal{I}(s, t, u; m_d)$  and the dimensionless variables  $s, t, u$  are invariant under U-duality. Therefore, the coefficients  $\mathcal{E}_{(m,n)}(m_d)$  must be automorphic forms on  $E_{d+1}(\mathbb{Z}) \backslash E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$ . The effective interactions produced by the lowest order terms  $\mathcal{E}_{(0,0)}(m_d)\mathcal{R}^4$ ,  $\mathcal{E}_{(1,0)}(m_d)D^4\mathcal{R}^4$ , and  $\mathcal{E}_{(0,1)}(m_d)D^6\mathcal{R}^4$  will be of central interest in this paper.

## 2.2 Perturbative contributions to the low energy expansion

Features of string perturbation theory, which are the concern of this paper, are obtained by expanding the coefficient functions  $\mathcal{E}_{(m,n)}(m_d)$  in powers of  $g_d$ . Each term in this expansion is then a function on the coset

$$\rho_d = G + B \in \text{SO}(d, d, \mathbb{R})/\text{SO}(d, \mathbb{R})^2 \quad (2.3)$$

parametrized by the metric  $G$  and two-form field  $B$  on  $\mathbb{T}^d$ , automorphic under the T-duality group  $\text{SO}(d, d, \mathbb{Z})$ . Mathematically, these perturbative contributions are obtained by extracting the constant term of the automorphic function  $\mathcal{E}_{(m,n)}$  with respect to the maximal parabolic sub-group  $P_d$  of  $E_{d+1}(\mathbb{R})$  whose Levi sub-group is  $L_d = \text{SO}(d, d, \mathbb{R}) \times \mathbb{R}^+$ , where the factor  $\mathbb{R}^+$  corresponds to the coupling  $g_d$ . In general however, this constant term may include non-analytic terms in  $g_d$ , such as powers of  $\log g_d$ . These terms do not exist in string perturbation theory, which is formulated by definition in the string frame, but may



arise when rescaling the amplitude to the Einstein frame, due to mixing between the local and non-local parts of the low-energy effective action [31, 33].

Finally, there are also exponentially suppressed contributions of order  $\mathcal{O}(e^{-2\pi/g_d})$  or smaller, with non-trivial dependence on the Ramond-Ramond moduli. These contributions are interpreted as D-brane instanton corrections (along with NS-brane instantons when  $d \geq 6$ ). Mathematically, they correspond to non-zero Fourier coefficients with respect to the unipotent radical  $P_d$ .

As a result, the structure of the perturbative expansion of the coefficients  $\mathcal{E}_{(m,n)}(m_d)$  of the higher-derivative local interactions takes the following general form,

$$\mathcal{E}_{(m,n)}(m_d) = g_d^{-\nu} \sum_{h=0}^{\infty} g_d^{-2+2h} \mathcal{E}_{(m,n)}^{(h)}(\rho_d) + \mathcal{E}_{(m,n)}^{\text{non-an.}}(g_d, \rho_d) + \mathcal{O}(e^{-2\pi/g_d}) \quad (2.4)$$

where the contribution  $\mathcal{E}_{(m,n)}^{(h)}(\rho_d)$  arises to  $h$ -loop order in superstring perturbation theory. The overall factor  $g_d^{-\nu}$ , with  $\nu = (2d - 4 + 8m + 12n)/(8 - d)$ , converts the genus  $h$  contribution from the string frame to the  $(10 - d)$ -dimensional Einstein frame. Our main focus in this work is on the perturbative contributions  $\mathcal{E}_{(m,n)}^{(h)}$ , but it is important to take into account the non-analytic contribution (when present) as it affects the differential equations satisfied by the functions  $\mathcal{E}_{(m,n)}^{(h)}$ .

The four-graviton scattering amplitude in superstring perturbation theory at arbitrary  $h$ -loop order involves an integral over the moduli space of super-Riemann surfaces of genus  $h$  with four punctures. In favorable cases (which include  $h = 1$  and  $h = 2$ ), this integral may be reduced to an integral over the moduli space  $\mathcal{M}_h$  of ordinary compact Riemann surfaces of genus  $h$ , and parametrized by the period matrix  $\Omega$  of the surface  $\Sigma$ , which takes values in the Siegel upper-half plane  $\mathcal{S}_h$  (subject, for  $h > 3$ , to Schottky relations). For the interactions  $\mathcal{E}_{(m,n)}$  of interest in this paper, the interplay between supersymmetry and dualities implies that the perturbative coefficients are non-zero for only a finite number of loop orders  $h$ , namely,

$$\begin{aligned} \mathcal{E}_{(0,0)}^{(h)}(\rho_d) &= 0 & h &\geq 2 \\ \mathcal{E}_{(1,0)}^{(h)}(\rho_d) &= 0 & h &\geq 3 \\ \mathcal{E}_{(0,1)}^{(h)}(\rho_d) &= 0 & h &\geq 4 \end{aligned} \quad (2.5)$$

Our conventions for the integration measure  $d\mu_h$  on  $\mathcal{M}_h$  are described in detail in appendix A. It is customary to express  $\Omega$  in terms of real matrices  $X, Y$  defined by  $\Omega = X + iY$ , and we shall do so also here throughout.

A key ingredient in genus  $h$  amplitudes in superstring theory compactification on a torus  $\mathbb{T}^d$  is the partition function for the zero-modes of the compact bosons on a genus  $h$  worldsheet,  $\Gamma_{d,d,h}(\rho_d; \Omega)$ , given by the following standard lattice sum,

$$\Gamma_{d,d,h}(\rho_d; \Omega) = (\det Y)^{d/2} \sum_{m_\alpha^I, n^{\alpha I} \in \mathbb{Z}} \exp \left\{ -\pi \mathcal{L}^{IJ}(\rho_d) Y_{IJ} + 2\pi i m_\alpha^I n^{\alpha J} X_{IJ} \right\} \quad (2.6)$$

where the quadratic form in  $m, n$  is defined by,

$$\mathcal{L}^{IJ}(\rho_d) = (m_\alpha^I + B_{\alpha\gamma} n^{\gamma I}) G^{\alpha\beta} (m_\beta^J + B_{\beta\delta} n^{\delta J}) + n^{\alpha I} G_{\alpha\beta} n^{\beta J} \quad (2.7)$$



The integers  $m_\alpha^I$  and  $n^{\alpha I}$  label momenta and windings. The range of the indices is  $I, J = 1, \dots, h$  and  $\alpha, \beta, \gamma, \delta = 1, \dots, d$ , and repeated indices are to be summed over. Note that we have  $\Gamma_{0,0,h} = 1$ .

In order to set the scene for the body of this paper, we list the results of explicit string perturbation theory calculations (a somewhat more detailed review is contained in [27]).

### 2.2.1 Genus zero

The tree-level amplitude can easily be expanded to all orders in the low energy expansion and the coefficients are independent of the moduli. Normalizing the classical tree-level term as in (2.2), the subsequent tree-level coefficients are given in terms of the Riemann zeta function  $\zeta(s)$  evaluated at odd integers by,

$$\mathcal{E}_{(0,0)}^{(0)}(\rho_d) = 2\zeta(3) \quad \mathcal{E}_{(1,0)}^{(0)}(\rho_d) = \zeta(5) \quad \mathcal{E}_{(0,1)}^{(0)}(\rho_d) = \frac{2}{3}\zeta(3)^2 \quad (2.8)$$

No dependence on the moduli  $\rho_d$  arises at tree-level since the momenta and polarization tensors of the four gravitons are along the subspace  $\mathbb{R}^{(10-d)}$ , but not along  $\mathbb{T}^d$ .

### 2.2.2 Genus one

The one-loop amplitude is an integral over the complex structure of the world-sheet torus and the contributions of terms in the low energy expansion reduce to integrals over the fundamental domain  $\mathcal{M}_1$  of the Poincaré upper half plane [18, 34], given by,

$$\mathcal{E}_{(0,0)}^{(1)}(\rho_d) = \pi \int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho_d; \tau) \quad (2.9)$$

$$\mathcal{E}_{(1,0)}^{(1)}(\rho_d) = 2\pi \int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho_d; \tau) E^*(2, \tau) \quad (2.10)$$

$$\mathcal{E}_{(0,1)}^{(1)}(\rho_d) = \frac{\pi}{3} \int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho_d; \tau) (5E^*(3, \tau) + \zeta(3)) \quad (2.11)$$

The modulus  $\tau$  parametrizes the genus-one moduli space  $\mathcal{M}_1$ , and its volume form  $d\mu_1$ , both of which are given in appendix A. The factor  $\Gamma_{d,d,1}(\rho_d; \tau)$  is the genus-one partition function on  $\mathbb{T}^d$  defined in (2.6) for general  $h$ . The quantity,

$$E^*(s, \tau) = \frac{1}{2} \pi^{-s} \Gamma(s) \zeta(2s) \sum_{(c,d)=1} \frac{(\text{Im } \tau)^s}{|c\tau + d|^{2s}} \quad (2.12)$$

is a non-holomorphic Eisenstein series, in the normalization of [35].

The integrals (2.9)–(2.11) are not generally convergent due to the polynomial growth of the integrand as  $\text{Im } \tau \rightarrow \infty$ . Specifically, the integral in (2.9) diverges for  $d \geq 2$ , while the integrals in (2.10) and (2.11) likewise diverge when  $d \geq 0$  since  $E^*(s, \tau) = \mathcal{O}((\text{Im } \tau)^{\max(s, 1-s)})$ . These divergences reflect the presence of a non-local term of the form  $s^{(D-8)/2} \mathcal{R}^4$ , (times  $\log s$  in even dimension  $D \geq 8$ ), produced by a one-loop infrared threshold, which dominates over the local term when  $D \leq 8$ . The prescription used in [18, 34] to separate the local and non-local contributions leads to a particular renormalization

prescription for these integrals, which is equivalent to one used in the Rankin-Selberg-Zagier method [35, 36]. In particular, for  $d = 0$ , the renormalized integral  $\int_{\mathcal{M}_1} d\mu_1 E^*(s, \tau)$  vanishes for any  $s$ , so that we have,

$$\mathcal{E}_{(0,0)}^{(1)} = 4\zeta(2) \quad \mathcal{E}_{(1,0)}^{(1)} = 0 \quad \mathcal{E}_{(0,1)}^{(1)} = \frac{4}{3}\zeta(2)\zeta(3) \quad (2.13)$$

We have suppressed the dependence on  $\rho_d$  since there are no such moduli for  $d = 0$ . For  $0 < d \leq 4$ , the result can instead be expressed in terms of the Eisenstein series  $E_{V,s}^{\text{SO}(d,d)}(\rho_d)$  associated with the parabolic subgroup of  $\text{SO}(d, d)$  that is labelled by the weight  $V$  of the vector representation of  $\text{SO}(d, d)$ ,

$$\mathcal{E}_{(0,0)}^{(1)}(\rho_d) = 2\pi^{2-\frac{d}{2}}\Gamma\left(\frac{d}{2}-1\right) E_{V,\frac{d}{2}-1}^{\text{SO}(d,d)}(\rho_d) \quad (d \neq 2) \quad (2.14)$$

$$\mathcal{E}_{(1,0)}^{(1)}(\rho_d) = \frac{2}{45}\pi^{2-\frac{d}{2}}\Gamma\left(1+\frac{d}{2}\right) E_{V,\frac{d}{2}+1}^{\text{SO}(d,d)}(\rho_d) \quad (d \neq 4) \quad (2.15)$$

$$\mathcal{E}_{(0,1)}^{(1)}(\rho_d) = \frac{\zeta(3)}{3}\mathcal{E}_{(0,0)}^{(1)}(\rho_d) + \frac{4}{567}\pi^{2-\frac{d}{2}}\Gamma\left(\frac{d}{2}+2\right) E_{V,\frac{d}{2}+2}^{\text{SO}(d,d)}(\rho_d) \quad (2.16)$$

The excluded values of  $d$  are those for which the integral is logarithmically divergent and the Eisenstein series has a pole. In that case, the formulae (2.14)–(2.16) hold after subtracting the pole.

### 2.2.3 Genus two

Since this is the case of central interest in this paper we will review it in somewhat more detail. The full two-loop four-graviton amplitude is given by [26],

$$\mathcal{A}_2^{(4)}(\epsilon_r, k_r; g_d, \rho_d) = \frac{\pi}{64}\kappa_d^2 g_s^2 \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}_2(s, t, u; \Omega) \Gamma_{d,d,2}(\rho_d; \Omega) \quad (2.17)$$

The reduced amplitude  $\mathcal{B}_2$  is given by,

$$\mathcal{B}_2(s, t, u; \Omega) = \int_{\Sigma^4} \frac{|\mathcal{Y}_S|^2}{(\det Y)^2} \exp \left\{ -\frac{\alpha'}{2} \sum_{i < j} k_i \cdot k_j G(z_i, z_j) \right\} \quad (2.18)$$

The integration is over four copies  $\Sigma^4$  of the genus-two Riemann surface  $\Sigma$ . The quantity  $\mathcal{Y}_S$  in the measure in (2.18) is a  $s, t, u$ -dependent family of holomorphic sections of the canonical line bundle  $K$  over  $\Sigma$  in each vertex insertion point  $z_i$  for  $i = 1, 2, 3, 4$ , as defined in [20]. The lattice partition function  $\Gamma_{d,d,2}$  was defined in (2.6).

The lowest order genus-two contribution  $\mathcal{E}_{(0,0)}^{(2)}(\rho_d)$  corresponds to the effective interaction  $\mathcal{R}^4$  and vanishes in any dimension [20],

$$\mathcal{E}_{(0,0)}^{(2)}(\rho_d) = 0 \quad (2.19)$$

The next contribution corresponds to  $D^4 \mathcal{R}^4$  and is given by [26],

$$\mathcal{E}_{(1,0)}^{(2)}(\rho_d) = \frac{\pi}{2} \int_{\mathcal{M}_2} d\mu_2 \Gamma_{d,d,2}(\rho_d; \Omega) \quad (2.20)$$

The integral is infrared divergent for  $d \geq 3$ , due to the presence of a two-loop non-local term of the form  $s^{D-7} D^4 \mathcal{R}^4$ , (times  $\log s$  in  $D = 7$ ), which dominates over the local term when  $D \leq 7$ . The renormalized integral can in principle be defined by a genus-two version of the Rankin-Selberg-Zagier prescription. The result may be expressed as the residue of the Langlands-Eisenstein series associated with the weight of the two-index antisymmetric representation of  $\mathrm{SO}(d, d)$  at  $s = d/2$  [37]. Alternatively, the conjectured results of [11], further supported in [12, 37], may be used to express the result as,

$$\mathcal{E}_{(1,0)}^{(2)}(\rho_d) = \frac{2}{3} \left( \hat{E}_{S,2}^{\mathrm{SO}(d,d)}(\rho_d) + \hat{E}_{C,2}^{\mathrm{SO}(d,d)}(\rho_d) \right) \quad (2.21)$$

where  $E_{S,s}^{\mathrm{SO}(d,d)}(\rho_d)$  and  $E_{C,s}^{\mathrm{SO}(d,d)}(\rho_d)$  are the Eisenstein series associated with the two distinct spinor weights  $S$  and  $C$  of  $\mathrm{SO}(d, d)$ , and the hat indicates that the simple pole at  $s = 2$  has been subtracted. For  $d = 0$ , one has,

$$\mathcal{E}_{(1,0)}^{(2)} = \frac{4}{3} \zeta(4) \quad (2.22)$$

in agreement with supersymmetry and S-duality, upon using the values of  $\zeta$  given in (A.7).

Our primary interest in this paper is the next term in the low energy expansion. It is the genus-two contribution to  $D^6 \mathcal{R}^4$  that was recently shown to have the form [27],

$$\mathcal{E}_{(0,1)}^{(2)}(\rho_d) = \pi \int_{\mathcal{M}_2} d\mu_2 \Gamma_{d,d,2}(\rho_d; \Omega) \varphi(\Omega) \quad (2.23)$$

where  $\varphi(\Omega)$  is the ZK invariant, whose form will be reviewed below. The integral over  $\mathcal{M}_2$  is convergent for  $d < 2$ , but has both primitive and one-loop subdivergences in  $d \geq 2$ . This is consistent with the presence of non-local terms of the form  $s^{(D-8)/2} D^6 \mathcal{R}^4$  and  $s^{D-8} D^6 R^4$  (times  $\log s$  when the power of  $s$  is integer). These contributions dominate over the local terms when  $D \leq 8$ . For  $d \geq 2$ , the integral must be renormalized, and it is not known at present how to express it in terms of Eisenstein series of  $\mathrm{SO}(d, d, \mathbb{Z})$ . For  $d = 0$  and  $d = 1$ , however, the values predicted by S-duality and supersymmetry are as follows [12, 17]

$$\mathcal{E}_{(0,1)}^{(2)} = \frac{8}{5} \zeta(2)^2 \quad (d = 0) \quad (2.24)$$

$$\mathcal{E}_{(0,1)}^{(2)}(\rho_d) = \frac{8}{5} \zeta(2)^2 \left( r^2 + \frac{1}{r^2} + \frac{5}{3} \right) \quad (d = 1) \quad (2.25)$$

where  $r$  is the radius of the  $d = 1$  circle in string units. The evaluation of the integral (2.23) for the case  $d = 0$  is the main focus of the subsequent sections of this paper.

### 2.2.4 Genus three

The three-loop contribution to  $D^6 \mathcal{R}^4$  in ten-dimensional type IIB string theory was recently computed in the pure spinor formalism [19], and claimed to take the value for  $d = 0$ ,

$$\mathcal{E}_{(0,1)}^{(3)} = \frac{4}{27} \zeta(6) \quad (2.26)$$

in agreement with the predictions from S-duality and supersymmetry [17]. However, at present there are unresolved issues concerning a factor of three in the derivation of this

value. Assuming the value is indeed correct, a straightforward generalization to the theory compactified on a torus  $\mathbb{T}^d$  leads to,

$$\mathcal{E}_{(0,1)}^{(3)}(\rho_d) = \frac{5}{16} \int_{\mathcal{M}_3} d\mu_3 \Gamma_{d,d,3}(\rho_d; \Omega) \quad (2.27)$$

where  $\mathcal{M}_3$  is the fundamental domain of genus-three Riemann surfaces, and  $\Gamma_{d,d,3}(\rho_d; \Omega)$  is the genus 3 lattice sum (2.6). Using the volume of  $\mathcal{M}_3$  stated in (A.8), it is easily seen that (2.27) reduces to (2.26) when  $d = 0$ . In general, the modular integral in (2.27) can be computed by a genus 3 extension of the Rankin-Selberg-Zagier method, and expressed as a residue of the Langlands-Eisenstein series associated with the weight of the three-index antisymmetric tensor representation of  $\text{SO}(d, d)$  at  $s = d/2$  [37]. Alternatively, using the result conjectured in [11] and further supported in [12], one has,

$$\mathcal{E}_{(0,1)}^{(3)}(\rho_d) = \frac{2}{27} \left( \hat{E}_{S,3}^{\text{SO}(d,d)}(\rho_d) + \hat{E}_{C,3}^{\text{SO}(d,d)}(\rho_d) \right) \quad (2.28)$$

and the hat indicates that the simple pole at  $s = 3$  has been subtracted.

### 2.3 S-duality and differential constraints

The three leading effective interactions in the  $\alpha'$  expansion of (2.2), namely  $\mathcal{R}^4$ ,  $D^4\mathcal{R}^4$ , and  $D^6\mathcal{R}^4$ , are BPS-saturated interactions. The exact coefficients  $\mathcal{E}_{(0,0)}(m_d)$  and  $\mathcal{E}_{(1,0)}(m_d)$  of the  $\mathcal{R}^4$  and  $D^4\mathcal{R}^4$ , including all non-perturbative corrections, have been conjectured based on variety of arguments including duality invariance, perturbative and decompactification limits, unitarity, and supersymmetry constraints.

For example, in ten-dimensional type IIB string theory ( $d = 0$ ) the coefficient  $\mathcal{E}_{(0,0)}(T)$  is proportional to the non-holomorphic  $\text{SL}(2, \mathbb{Z})$  Eisenstein series  $E^*(\frac{3}{2}, T)$  [6]. It is the unique solution of the Laplace equation  $(\Delta_{\text{SL}(2)} - \frac{3}{4})\mathcal{E}_{(0,0)}(T) = 0$  on the upper half  $T$  plane, a constraint which follows from a careful implementation of nonlinear extended supersymmetry [7, 8]. This also extends to  $\mathcal{E}_{(1,0)}(T)$  in  $d = 0$ , which is proportional to the Eisenstein series  $E^*(\frac{5}{2}, T)$  [8, 10]. More generally, the conjectured coefficients  $\mathcal{E}_{(0,0)}(m_d)$  and  $\mathcal{E}_{(1,0)}(m_d)$  are given by linear combinations of Eisenstein series (and derivatives thereof) under the duality group  $E_{d+1}(\mathbb{Z})$ . They satisfy Laplace eigenvalue equations [6, 7, 9, 11, 12, 17],

$$\left( \Delta_{E_{d+1}} - \frac{3(d+1)(2-d)}{(8-d)} \right) \mathcal{E}_{(0,0)}(m_d) = 6\pi \delta_{d,2} \quad (2.29)$$

$$\left( \Delta_{E_{d+1}} - \frac{5(d+2)(3-d)}{(8-d)} \right) \mathcal{E}_{(1,0)}(m_d) = 40 \zeta(2) \delta_{d,3} + 7 \mathcal{E}_{(0,0)} \delta_{d,4} \quad (2.30)$$

where<sup>5</sup>  $\Delta_{E_{d+1}}$  is the Laplace-Beltrami operator on the moduli space  $E_{d+1}(\mathbb{R})/K_{d+1}(\mathbb{R})$ . It is expected that these equations are consequences of non-linear supersymmetry, although this has not been fully established yet. The anomalous terms on the right-hand sides occur in dimensions in which the eigenvalues vanish, or when the eigenvalue of  $\mathcal{E}_{(1,0)}$  becomes

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<sup>5</sup>Our normalization for the Laplace-Beltrami operators  $\Delta_{E_{d+1}}$  and  $\Delta_{\text{SO}(d,d)}$  differ by a factor of 2 from the ones used in [11, 35, 37, 39].

degenerate with that of  $\mathcal{E}_{(0,0)}$ . These terms are correlated with logarithmic infrared divergences in string theory and in supergravity. Consequently, they are also correlated with the onset of ultraviolet divergences in supergravity [33].

For the  $D^6\mathcal{R}^4$  interaction, a candidate for the exact coefficient  $\mathcal{E}_{(0,1)}(m_d)$  is only available in dimensions  $d = 0, 1, 2$ , and in a rather implicit form [12, 17, 38]. General arguments, supergravity limits, and many consistency checks suggest that  $\mathcal{E}_{(0,1)}(m_d)$  is a solution of the inhomogeneous Laplace eigenvalue equation,<sup>6</sup>

$$\left(\Delta_{E_{d+1}} - \frac{6(4-d)(d+4)}{8-d}\right) \mathcal{E}_{(0,1)}(m_d) = -(\mathcal{E}_{(0,0)}(m_d))^2 + 40\zeta(3)\delta_{d,4} \quad (2.31)$$

The quadratic term on the r.h.s. can be understood qualitatively [17] as a consequence of the  $(\alpha')^3$  corrections to the supersymmetry variations, although there has not been a precise derivation of this equation based on supersymmetry. The other term is an anomalous contribution which arises when the eigenvalue of  $\mathcal{E}_{(0,1)}$  vanishes. They reflect the occurrence of logarithmic infrared divergences in string theory. The exact solutions to (2.31) relevant for  $\mathcal{E}_{(0,1)}$  are not known explicitly, but the perturbative expansions are, as reviewed in the previous subsection.

### 2.3.1 S-duality constraints on the perturbative coefficients

Having summarized the differential constraints satisfied by the exact coefficients  $\mathcal{E}_{(m,n)}(m_d)$  of the  $\mathcal{R}^4$ ,  $D^4\mathcal{R}^4$  and  $D^6\mathcal{R}^4$  interactions in the low energy expansion, it is now in principle straightforward to determine the differential constraints satisfied by the perturbative terms  $\mathcal{E}_{(m,n)}^{(h)}(\rho_d)$  in the weak coupling expansion (2.4). For this purpose, however, it is important to include the contribution  $\mathcal{E}_{(m,n)}^{\text{non-an.}}(g_d, \rho_d)$  of the terms proportional to powers of  $\log g_d$ . These terms can be computed from the putative exact result, or from the non-local terms in the supergravity amplitude, and are recorded below,<sup>7</sup>

$$\begin{aligned} \mathcal{E}_{(0,0)}^{\text{non-an.}}(g_d, \rho_d) &= \frac{4\pi}{3} \ln g_2 \delta_{d,2} \\ \mathcal{E}_{(1,0)}^{\text{non-an.}}(g_d, \rho_d) &= \frac{16\pi^2}{15} \ln g_3 \delta_{d,3} + \mathcal{E}_{(0,0)} \ln g_4 \delta_{d,4} \\ \mathcal{E}_{(0,1)}^{\text{non-an.}}(g_d, \rho_d) &= \left(-\frac{4\pi^2}{27} \ln^2 g_2 + \frac{2\pi}{9} \left(\frac{\pi}{2} + \mathcal{E}_{(0,0)}\right) \ln g_2\right) \delta_{d,2} + 5\zeta(3) \ln g_4 \delta_{d,4} \end{aligned} \quad (2.32)$$

The action of the Laplacian  $\Delta_{E_{d+1}}$  on  $\mathcal{E}_{(m,n)}$  can now be decomposed terms of the  $\text{SO}(d, d)$  subgroup of  $E_{d+1}$ ,

$$\Delta_{E_{d+1}} = \frac{8-d}{8} \partial_\phi^2 + \frac{d^2-d+4}{4} \partial_\phi + \Delta_{\text{SO}(d,d)} + \dots \quad (2.33)$$

<sup>6</sup>The  $\delta_{d,4}$  terms in (2.30) and (2.31) have been corrected from the ones given in equations (2.8) and (2.9) in [33].

<sup>7</sup>The coefficient of  $\ln g_4$  in the last line agrees with the coefficient of the infrared singularity of the  $D = 6$  three-loop supergravity amplitude computed in [32], thereby resolving a puzzle raised following equation (3.21) in [33].

where we have set  $g_d = e^\phi$ , and the ellipsis denotes derivatives with respect to the Ramond-Ramond moduli which are angular variables that decouple from perturbation theory. This agrees with [11, A.24] for  $d = 2, 3, 4$ , upon noting that for these values,

$$2^{d-1} = \frac{2(d^2 - d + 4)}{8 - d} \quad (2.34)$$

Substituting (2.33) into (2.29)–(2.31), we deduce that the  $h$ -loop contributions to the coefficients  $\mathcal{E}_{(m,m)}$  satisfy the following Laplace-type equations:

- The perturbative parts of  $\mathcal{E}_{(0,0)}(m_d)$  satisfy

$$\begin{aligned} \Delta_{\text{SO}(d,d)} \mathcal{E}_{(0,0)}^{(0)}(\rho_d) &= 0 \\ (\Delta_{\text{SO}(d,d)} + d(d-2)/2) \mathcal{E}_{(0,0)}^{(1)}(\rho_d) &= 4\pi \delta_{d,2} \end{aligned} \quad (2.35)$$

- The perturbative parts of  $\mathcal{E}_{(1,0)}(m_d)$  satisfy

$$\begin{aligned} \Delta_{\text{SO}(d,d)} \mathcal{E}_{(1,0)}^{(0)}(\rho_d) &= 0 \\ (\Delta_{\text{SO}(d,d)} + (d+2)(d-4)/2) \mathcal{E}_{(1,0)}^{(1)}(\rho_d) &= 12\zeta(3) \delta_{d,4} \\ (\Delta_{\text{SO}(d,d)} + d(d-3)) \mathcal{E}_{(1,0)}^{(2)}(\rho_d) &= 24\zeta(2) \delta_{d,3} + 4\mathcal{E}_{(0,0)}^{(1)}(\rho_d) \delta_{d,4} \end{aligned} \quad (2.36)$$

- The perturbative parts of  $\mathcal{E}_{(0,1)}(m_d)$  satisfy

$$\begin{aligned} (\Delta_{\text{SO}(d,d)} - 6) \mathcal{E}_{(0,1)}^{(0)}(\rho_d) &= -\left(\mathcal{E}_{(0,0)}^{(0)}(\rho_d)\right)^2 \\ (\Delta_{\text{SO}(d,d)} - (d+4)(6-d)/2) \mathcal{E}_{(0,1)}^{(1)}(\rho_d) &= -2\mathcal{E}_{(0,0)}^{(0)}(\rho_d) \mathcal{E}_{(0,0)}^{(1)}(\rho_d) + \frac{2\pi}{3}\zeta(3) \delta_{d,2} \\ (\Delta_{\text{SO}(d,d)} - (d+2)(5-d)) \mathcal{E}_{(0,1)}^{(2)}(\rho_d) &= -\left(\mathcal{E}_{(0,0)}^{(1)}(\rho_d)\right)^2 - \left(\frac{\pi}{3}\mathcal{E}_{(0,0)}^{(1)} + \frac{7\pi^2}{18}\right) \delta_{d,2} \\ (\Delta_{\text{SO}(d,d)} - 3d(4-d)/2) \mathcal{E}_{(0,1)}^{(3)}(\rho_d) &= 20\zeta(3) \delta_{d,4} \end{aligned} \quad (2.37)$$

For all but the genus two  $D^6\mathcal{R}^4$  amplitude  $\mathcal{E}_{(0,1)}^{(2)}(\rho_d)$  it is relatively straightforward to check that these equations are consistent with the Eisenstein series which describe the perturbative terms, as discussed in section 2.2. In particular, the anomalous terms on the r.h.s. arise whenever the Eisenstein series has a pole, after subtracting the contribution of the pole. In the case  $d = 0$ , where the coefficients are constants and the Laplacian  $\Delta_{\text{SO}(d,d)}$  vanishes, no such anomalous terms arise and these equations reduce to algebraic relations between the coefficients. In particular, the two-loop contribution  $\mathcal{E}_{(0,1)}^{(2)}(\rho_d)$  is predicted to take the value stated in (2.24).

In the sequel we shall compute the genus-two modular integral (2.23) for  $d = 0$  and check agreement with this prediction. The differential equation on the third line of (2.38) will indicate that the Zhang-Kawazumi invariant satisfies the differential equation (1.4), which holds the key to the computation of the modular integral.

### 3 The Zhang-Kawazumi invariant

The ZK invariant  $\varphi(\Sigma)$  may be defined on a surface  $\Sigma$  of arbitrary genus  $h \geq 2$  [28, 29]. As was already stated in the Introduction (1.3), it was shown in [27] to be given by the following expression  $\varphi(\Sigma) = -\int_{\Sigma^2} P(x, y) G(x, y)/4h$ . Here, the bi-form  $P(x, y)$  is a section of  $K \otimes \bar{K}$  in both  $x$  and  $y$ , where  $K$  is the canonical bundle on  $\Sigma$ , and may be expressed as,

$$P(x, y) = \sum_{I, J, K, L} P_{IJKL} \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

$$P_{IJKL} = -Y_{IJ}^{-1} Y_{KL}^{-1} + h Y_{IL}^{-1} Y_{JK}^{-1} \quad (3.1)$$

The indices take the values  $I, J, K, L = 1, \dots, h$ ; henceforth, repeated indices will be understood to be summed. Furthermore,  $\omega_I$  is a basis of canonically normalized holomorphic Abelian differentials, and  $\Omega_{IJ} = X_{IJ} + iY_{IJ}$  is the period matrix of the surface  $\Sigma$  (see appendix A for their detailed definitions.). The scalar Green function  $G$  on  $\Sigma^2$  is given in terms of the prime form,  $E(x, y)$ , and the above quantities by,

$$G(x, y) = -\ln |E(x, y)|^2 + 2\pi Y_{IJ}^{-1} \left( \text{Im} \int_x^y \omega_I \right) \left( \text{Im} \int_x^y \omega_J \right) \quad (3.2)$$

Its mixed derivatives are given by<sup>8</sup>

$$\begin{aligned} \partial_{\bar{y}} \partial_x G(x, y) &= +2\pi \delta(x, y) - \pi Y_{IJ}^{-1} \omega_I(x) \overline{\omega_J(y)} \\ \partial_{\bar{y}} \partial_y G(x, y) &= -2\pi \delta(x, y) + \pi Y_{IJ}^{-1} \omega_I(y) \overline{\omega_J(y)} \end{aligned} \quad (3.3)$$

The bi-form  $P(x, y)$  is symmetric under the interchange of  $x$  and  $y$ , and obeys the key property that its integral over a single copy of  $\Sigma$  vanishes identically,

$$\int_{\Sigma_x} P(x, y) = 0 \quad (3.4)$$

The formula results from the combination of the following two elementary results,

$$\sum_{KL} P_{IJKL} Y_{KL} = 0, \quad \int_{\Sigma} \omega_K \wedge \overline{\omega_L} = -2i Y_{KL} \quad (3.5)$$

The Green function is single-valued but transforms under conformal transformations in  $x$  by a shift which depends only upon  $x$ . Combining this transformation with the property of (3.4) guarantees that  $\varphi(\Omega)$  in (1.3) is well-defined. In fact, in view of (3.4), any properly normalized scalar Green function may be used instead of  $G$ , including the properly normalized Arakelov Green function (see [27] for the detailed relations). Finally, we shall often write  $\varphi(\Omega)$  instead of  $\varphi(\Sigma)$  when we parametrize  $\Sigma$  by its period matrix, by a slight abuse of notation.

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<sup>8</sup>Depending on context, we write  $\omega_I$  for the differential one form, or  $\omega_I(z)$  for the function representing the 1-form in a local coordinate system  $(z, \bar{z})$ , so that the 1-form locally takes the form  $\omega_I = \omega_I(z) dz$ . In the latter case, the convention for the integral over  $\Sigma$  includes a factor of the volume form  $idz \wedge d\bar{z}$ , which will not, however, be exhibited.



### 3.1 Differential constraint on the ZK invariant: a first hint

In order to gain insight into the nature of the Zhang-Kawazumi invariant  $\varphi$ , it is very instructive to examine the structure of the differential equation satisfied by the genus-two coefficient  $\mathcal{E}_{(0,1)}^{(2)}$  defined by the modular integral (2.23). On the one hand, this coefficient satisfies the equation in the third line of (3.6), which we reproduce below for  $d \neq 2$ ,

$$(\Delta_{\text{SO}(d,d)} - (d+2)(5-d)) \mathcal{E}_{(0,1)}^{(2)}(\rho_d) = -\left(\mathcal{E}_{(0,0)}^{(1)}(\rho_d)\right)^2 \quad (3.6)$$

On the other hand, the lattice partition function for the torus  $\mathbb{T}^d$  satisfies the following differential equation in arbitrary genus  $h$  [11],

$$\left(\Delta_{\text{SO}(d,d)} - 2\Delta_{\text{Sp}(2h)} + \frac{1}{2}dh(d-h-1)\right) \Gamma_{d,d,h}(\rho_d; \Omega) = 0 \quad (3.7)$$

where  $\Delta_{\text{Sp}(2h)}$  is the Laplacian on the Siegel upper-half space  $\mathcal{S}_h$ , defined in appendix A. Specializing to genus-two, where the moduli space  $\mathcal{M}_2$  coincides with the Siegel upper half space  $\mathcal{S}_2$ , and the Laplacian  $\Delta$  coincides with  $\Delta_{\text{Sp}(4)}$ , it follows from (2.23) and (3.7) that

$$(\Delta_{\text{SO}(d,d)} - (d+2)(5-d)) \mathcal{E}_{(0,1)}^{(2)}(\rho_d) = 2\pi \int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega) (\Delta - 5) \Gamma_{d,d,2}(\rho_d, \Omega) \quad (3.8)$$

Comparing with the differential equation (3.6), we see that agreement for  $d \neq 2$  requires,

$$\int_{\mathcal{M}_2} d\mu_2 \varphi(\Omega) (\Delta - 5) \Gamma_{d,d,2}(\rho_d, \Omega) = -\frac{\pi}{2} \left( \int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho_d, \tau) \right)^2 \quad (3.9)$$

After integration by parts, this becomes,

$$\begin{aligned} \int_{\mathcal{M}_2} d\mu_2 \Gamma_{d,d,2}(\rho_d, \Omega) (\Delta - 5) \varphi(\Omega) + \int_{\partial\mathcal{M}_2} (\varphi \star d\Gamma_{d,d,2} - \Gamma_{d,d,2} \star d\varphi) \\ = -\frac{\pi}{2} \left( \int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho_d, \tau) \right)^2 \end{aligned} \quad (3.10)$$

The structure of this equation is very informative. Recall that the boundary of  $\mathcal{M}_2$  includes the separating degeneration limit,  $\Omega_{12} \rightarrow 0$ , where  $\Gamma_{d,d,2}(\rho_d; \Omega) \sim \Gamma_{d,d,1}(\rho_d; \Omega_{11}) \Gamma_{d,d,1}(\rho_d; \Omega_{22})$  has the factorised form of the right-hand side of the equation. This is the first indication that the combination  $(\Delta - 5) \varphi$  has support on boundary  $\partial\mathcal{M}_2$  of moduli space.

### 3.2 The ZK invariant in the supergravity limit

Further evidence in support of the eigenvalue equation  $(\Delta - 5) \varphi = 0$  in the interior of  $\mathcal{M}_2$  may be gathered by considering the limit of degenerating Riemann surfaces  $\Sigma$ . The Deligne-Mumford compactification of  $\mathcal{M}_2$  requires the addition to  $\mathcal{M}_2$  of just two divisors, namely the separating and the non-separating nodes. These nodes intersect, and contain further degeneration divisors. One might study the fate of the equation  $(\Delta - 5) \varphi = 0$  on any of these degenerations.

Here, we shall limit attention to studying the complete non-separating degeneration in which the components of  $Y = \text{Im } \Omega$  all become large, and the surface degenerates

to two connected long thin tubes. This degeneration is physically significant, since it is directly related to the integrand of two-loop Feynman diagrams in  $D = 10$  supergravity. In particular the four-graviton amplitude in maximally supersymmetric theories can be rewritten in terms of graphs with cubic vertices. Ignoring the position of the external gravitons one obtains a skeleton graph with two tri-valent vertices. The lengths of the corresponding lines of the graph will be denoted by  $L_i \gg 1$ , and may be identified with the entries of  $Y$  as follows,

$$\Omega = iY_L + \mathcal{O}(1) \qquad Y_L = \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix} \qquad (3.11)$$

The  $\mathcal{O}(1)$  corrections which are being omitted here contain both the real parts of  $\Omega$  as well as higher order corrections to  $\text{Im } \Omega$ . To leading order in  $L_i \gg 1$ , the complete degeneration limit of the integral (1.3) may be expressed in terms of the graph lengths  $L_i$  and the positions,  $L_x$  and  $L_y$ , of the points  $x, y$  which enter into the integral. The limiting behavior of Abelian differentials and of the prime form are standard, and have been discussed with the help of the Schottky parametrisation in [40], as well as in the context of tropical modular geometry in [41]. For our purposes it is sufficient to know the limits of the prime form and the Abelian differentials:  $E(x, y)$  tends to the distance on the graph between the two insertion points multiplied (in our conventions) by  $2\pi$ , while  $\text{Im } \omega_1(x) = dL_x$  (respectively  $\text{Im } \omega_2(x) = dL_x$ ) if  $x$  is on the thin tubes forming the first (respectively second) loop and zero otherwise.

The contribution to (1.3) from the complete non-separating degeneration arises from two graph topologies: type (a) where the insertions  $x, y$  are on the same degenerating tube; and type (b) when they are on opposite tubes. The corresponding asymptotics of the Green function  $G$  for both graphs were obtained in [42], and are given by,<sup>9</sup>

$$\begin{aligned} G^{(a)}(x, y) &\rightarrow G_L^{(a)} = -2\pi \left( |L_x - L_y| - \frac{(L_2 + L_3)(L_x - L_y)^2}{\det Y_L} \right) \\ G^{(b)}(x, y) &\rightarrow G_L^{(b)} = -2\pi \left( L_x + L_y - \frac{(L_1 + L_3)L_y^2 + (L_2 + L_3)L_x^2 + 2L_x L_y L_3}{\det Y_L} \right) \end{aligned} \qquad (3.12)$$

These formulas have been written down for type (a) when the points  $x, y$  are on the tubes of length  $L_1$ ; and for type (b) when  $x$  is on the tube of length  $L_1$  while  $y$  is on the tube of length  $L_2$ . In the same limit the bi-form  $P$  of (3.1) on  $\Sigma^2$  for both types of graphs takes the following form,

$$\begin{aligned} P^{(a)}(x, y) &\rightarrow P_L^{(a)} dL_x dL_y = -4 \left( \frac{L_2 + L_3}{\det Y_L} \right)^2 dL_x dL_y \\ P^{(b)}(x, y) &\rightarrow P_L^{(b)} dL_x dL_y = -4 \left( \frac{L_3^2 - \det Y_L}{\det Y_L^2} \right) dL_x dL_y \end{aligned} \qquad (3.13)$$

under the same assumptions on  $x, y$  as we had spelled out for the Green function.

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<sup>9</sup>Here and below, the arrows encompass both taking the limit of large  $L_i \gg 1$ , as well as carrying out the angular integrations on moduli and the points  $x, y$ .

For both types of graph an overall factor of 2 arises from the possibility of exchanging the ordering of the two points  $x, y$ . Then there are other contributions with the same topology which are simply obtained by cyclical permutation of the  $L_i$  and correspond to inserting the punctures on the other tubes forming the genus 2 degenerating surface. Thus from the diagrams of type (a) we have

$$\varphi_L^{(a)} = -\frac{1}{4} \int_0^{L_1} dL_x \int_0^{L_x} dL_y P_L^{(a)} G_L^{(a)} + \text{cycl.} = 2\pi \left[ -\frac{L_\Sigma}{12} + \frac{L_1 L_2 L_3}{4 \det Y_L} + \frac{L_1^2 L_2^2 L_3^2 L_\Sigma}{6 \det Y_L^3} \right] \quad (3.14)$$

where  $L_\Sigma = L_1 + L_2 + L_3$ . Similarly the contributions of the diagrams of type (b) is

$$\varphi_L^{(b)} = -\frac{1}{4} \int_0^{L_1} dL_x \int_0^{L_2} dL_y P_L^{(b)} G_L^{(b)} + \text{cycl.} = 2\pi \left[ \frac{L_\Sigma}{6} - \frac{2L_1 L_2 L_3}{3 \det Y_L} - \frac{L_1^2 L_2^2 L_3^2 L_\Sigma}{6 \det Y_L^3} \right] \quad (3.15)$$

Adding up the two types of contributions  $\varphi_L^{(a)} + \varphi_L^{(b)} = \varphi_L$  gives the following expression for  $\varphi$  in the complete degeneration limit,

$$\varphi(\Sigma) = \varphi_L + \mathcal{O}(1) \quad \varphi_L = \frac{\pi}{6} \left( L_1 + L_2 + L_3 - \frac{5 L_1 L_2 L_3}{L_1 L_2 + L_1 L_3 + L_2 L_3} \right) \quad (3.16)$$

The contribution from the complete non-separating degeneration to the two-loop  $D^6 \mathcal{R}^4$  effective interaction is therefore given by inserting the asymptotic expression (3.16) for  $\varphi$  into the integral (2.23), and setting the winding numbers  $n^{aI}$  in the lattice sum (2.6) to zero.

The result should match the two-loop  $D^6 \mathcal{R}^4$  effective interaction in 10-dimensional supergravity compactified on  $\mathbb{T}^d$ , which differs by a factor of  $(\det Y)^{1/2}$  from the same interaction computed in 11-dimensional supergravity compactified on  $\mathbb{T}^d$ . The latter was computed in equation (3.6) of [17] (or in equation (2.23) of [42] after correcting a sign), and is in agreement with (3.16). The asymptotic expression (3.16), in the limit where  $L_2 \gg L_1, L_3$ , is also in agreement with the (double) degeneration limit  $\tau, \tau_1 \rightarrow i\infty$  of the non-separating degeneration formula (5.4), which is already an expansion for  $\tau_2 \rightarrow i\infty$ .

To verify the equation  $(\Delta - 5)\varphi = 0$  in the complete degeneration limit, it remains to evaluate the Laplacian  $\Delta_L$  of (A.9) in this limit and we find,

$$\Delta_L = \sum_{i,j} L_i L_j \frac{\partial}{\partial L_i} \frac{\partial}{\partial L_j} + \frac{\det Y_L}{2} \left( \sum_{i=1}^3 \frac{\partial^2}{\partial L_i^2} - 2 \sum_{i < j} \frac{\partial}{\partial L_i} \frac{\partial}{\partial L_j} \right) \quad (3.17)$$

It is easy to check that we indeed have  $(\Delta_L - 5)\varphi_L = 0$ . With this additional encouragement we will now proceed to show that  $\varphi$  does indeed satisfy (1.4) for all genus-two surfaces  $\Sigma$ .

#### 4 The Laplacian of the Zhang-Kawazumi invariant

The purpose of this section is to calculate, from first principles, the Laplacian of the ZK invariant  $\varphi$ , and show that  $\varphi$  obeys the eigenvalue equation  $(\Delta - 5)\varphi = 0$  in the interior of

the moduli space of genus-two Riemann surfaces  $\mathcal{M}_2$ . The equation may be extended to the Deligne-Mumford compactification  $\overline{\mathcal{M}}_2$  at the cost of a right side which has support on the separating node. The derivation in this section is somewhat technical and the hurried reader may wish to skip to section 5.

## 4.1 Preliminaries

The Laplacian of  $\varphi$  will be computed using standard deformation theory of complex structures on a Riemann surface. As it turns out, the problem may be formulated in arbitrary genus with little extra complication, and we shall carry out the calculations in arbitrary genus. A brief summary of the Siegel upper half space  $\mathcal{S}_h$  for arbitrary genus  $h$ , its Poincaré metric, volume form, action of the modular group, and sub-variety of the moduli space  $\mathcal{M}_h$  of compact genus  $h$  Riemann surfaces is provided in appendix A.

The  $\mathrm{Sp}(2h, \mathbb{R})$ -invariant Laplace operator  $\Delta_{\mathrm{Sp}(2h)}$  on scalar functions on  $\mathcal{S}_h$  is defined in (A.9). For  $h = 2, 3$  the  $\mathrm{Sp}(2h, \mathbb{R})$ -invariance of the Laplacian  $\Delta_{\mathrm{Sp}(2h)}$  automatically induces a Laplacian on  $\mathcal{M}_2$  which we shall denote  $\Delta$ . For  $h \geq 4$ , the Laplacian  $\Delta_{\mathrm{Sp}(2h)}$  needs to be projected by restricting the derivatives  $\partial_{IJ}$  in (A.9) to the tangent space  $T\mathcal{M}_h$  at every point of  $\mathcal{M}_h$ , and we shall denote the resulting Laplacian by  $\Delta$ .

## 4.2 Basic variational formulas

To evaluate the derivatives with respect to  $\Omega$  and  $\bar{\Omega}$  (projected onto  $T\mathcal{M}_h$  for  $h \geq 4$ ), we shall use the standard theory and formulas of deformations of complex structures. The tangent space  $T\mathcal{M}_h$  decomposes into a direct sum of holomorphic and anti-holomorphic subspaces, which are generated respectively by a Beltrami differential  $\mu$  and its complex conjugate  $\bar{\mu}$ . Here,  $\mu$  is a section of  $K^{-1} \otimes \bar{K}$  where  $K$  is the canonical bundle on  $\Sigma$ . To evaluate the Laplacian  $\Delta$  on  $\varphi$ , we shall need to compute the mixed variational derivatives of  $\varphi$  with respect to  $\mu$  and  $\bar{\mu}$ , which automatically includes the needed projection from  $T\mathcal{S}_h$  to  $T\mathcal{M}_h$ .

A holomorphic deformation  $\delta_\mu \phi$  with Beltrami differential  $\mu = \mu_{\bar{w}}^w d\bar{w}/dw$  of any function  $\phi$  on  $\mathcal{M}_h$  is given as follows,

$$\delta_\mu \phi = \frac{1}{2\pi} \int_{\Sigma} d^2w \mu_{\bar{w}}^w \delta_{ww} \phi \quad (4.1)$$

The deformations  $\delta_{ww} \phi$  supported at the point  $w$  may be viewed as resulting from the insertion of the stress tensor at the point  $w$ , and the particular normalization used here is in accord with the standard normalizations of the stress tensor [43].

The point-wise deformation of the period matrix, the canonically normalized holomorphic Abelian differentials  $\omega_I(x)$ , and the prime form  $E(x, y)$  are given as follows [44],

$$\begin{aligned} \delta_{ww} \omega_I(x) &= \omega_I(w) \partial_x \partial_w \ln E(x, w) \\ \delta_{ww} \Omega_{IJ} &= 2\pi i \omega_I(w) \omega_J(w) \\ \delta_{ww} \ln E(x, y) &= -\frac{1}{2} \left( \partial_w \ln E(w, x) - \partial_w \ln E(w, y) \right)^2 \end{aligned} \quad (4.2)$$

The deformation of other quantities, such as Abelian differentials of the second and third kind, may be obtained from the last equation by taking derivatives in  $x$  and  $y$ .

### 4.3 Calculation of the first variational derivative

From the basic point-wise variational formulas of (4.2), we now produce further variational formulas which will be of more direct utility in evaluating the Laplacian of  $\varphi$ . We shall prefer to express the resulting formulas in terms of the single-valued Green function  $G$  rather than in terms of the multiple-valued prime form, and Abelian integrals. First, one derives the following variational formulas for  $G$  and  $P$ ,

$$\begin{aligned}\delta_{ww}G(x, y) &= \frac{1}{2} \left( \partial_w G(w, x) - \partial_w G(w, y) \right)^2 \\ \delta_{ww}P(x, y) &= -P_{IJKL} \omega_I(w) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)} \partial_x \partial_w G(w, x) \\ &\quad - P_{IJKL} \omega_I(x) \overline{\omega_J(x)} \omega_K(w) \overline{\omega_L(y)} \partial_y \partial_w G(w, y)\end{aligned}\quad (4.3)$$

where  $P_{IJKL}$  is the modular tensor defined in (3.1). A useful intermediate formula in the derivation of both formulas in (4.3) is given by the relation,

$$\delta_{ww} (Y_{IJ}^{-1} \omega_J(x)) = -Y_{IJ}^{-1} \omega_J(w) \partial_x \partial_w G(w, x) \quad (4.4)$$

With the help of these formulas, we obtain the first order variation of  $\varphi$ ,

$$\begin{aligned}4h \delta_{ww} \varphi &= \int_{\Sigma^2} \left\{ P_{IJKL} \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)} \partial_w G(w, x) \partial_w G(w, y) \right. \\ &\quad - P_{IJKL} \omega_I(w) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)} \partial_w G(w, x) \partial_x G(x, y) \\ &\quad \left. - P_{IJKL} \omega_I(x) \overline{\omega_J(x)} \omega_K(w) \overline{\omega_L(y)} \partial_w G(w, y) \partial_y G(x, y) \right\} \quad (4.5)\end{aligned}$$

Note that the terms proportional to  $(\partial_w G(w, x))^2$  and  $(\partial_w G(w, y))^2$ , which arise from the variation of  $G$  in the first line of (4.3), cancel in view of (3.4). Upon interchange of  $x$  and  $y$ , the last two terms above are seen to be equal; we have refrained from carrying through the corresponding simplification in order to retain the manifest symmetry under interchange of  $x$  and  $y$ . With the help of the mixed derivative formulas for  $G$  in (3.3), one readily verifies holomorphicity of the first variation, namely  $\partial_{\bar{w}} (\delta_{ww} \varphi) = 0$ .

### 4.4 Calculation of the second variational derivative

To compute the mixed variation  $\delta_{\bar{u}\bar{u}} \delta_{ww} \varphi$ , we use the complex conjugated relations of (4.2), but we also need to vary the derivatives with respect to the holomorphic coordinates. The starting point to do so are the standard variational formulas for the Cauchy-Riemann operators  $\partial_z^{(n)}$  and  $\partial_{\bar{z}}^{(n)}$  on sections of  $K^n$ , which are given by [43],

$$\delta_{\bar{\mu}} \partial_{\bar{z}}^{(n)} = 0 \quad \delta_{\bar{\mu}} \partial_z^{(n)} = \bar{\mu} \partial_{\bar{z}}^{(n)} + n (\partial_{\bar{z}} \bar{\mu}) \quad (4.6)$$

Here, we set  $n = 0$ , drop the superscript  $(n)$ , and derive the point-wise deformations,

$$\delta_{\bar{u}\bar{u}} \partial_{\bar{z}} = 0 \quad \delta_{\bar{u}\bar{u}} \partial_z = 2\pi \delta(z, u) \partial_{\bar{z}} \quad (4.7)$$

The key ingredients needed for the variation of (4.5) include the variation of a single derivative of  $G$ , which is found to be,

$$\delta_{\bar{u}\bar{u}}\left(\partial_x G(x, y)\right) = -\pi\partial_{\bar{x}}\delta(u, x) + \pi Y_{IJ}^{-1}\omega_I(x)\overline{\omega_J(u)}\left(\partial_{\bar{u}}G(u, y) - \partial_{\bar{u}}G(u, x)\right) \quad (4.8)$$

and the variation of the generalization of  $P(x, y)$  which appears in (4.5), and which takes the following form,

$$\begin{aligned} \delta_{\bar{u}\bar{u}}\left(P_{IJKL}\omega_I(s)\overline{\omega_J(x)}\omega_K(t)\overline{\omega_L(y)}\right) = & -P_{IJKL}\omega_I(s)\overline{\omega_J(u)}\omega_K(t)\overline{\omega_L(y)}\partial_{\bar{u}}\partial_{\bar{x}}G(u, x) \\ & -P_{IJKL}\omega_I(s)\overline{\omega_J(x)}\omega_K(t)\overline{\omega_L(u)}\partial_{\bar{u}}\partial_{\bar{y}}G(u, y) \end{aligned} \quad (4.9)$$

for arbitrary points  $s, t, x, y \in \Sigma$ .

The calculation of these variational derivatives is fairly lengthy, and is relegated to appendix B. The final result, valid for arbitrary genus, may be cast in the following form,

$$\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi = \psi_A + \psi_B + \psi_C \quad (4.10)$$

where each one of these contributions is given by,

$$\begin{aligned} \psi_A = & -\frac{2\pi}{4h}(2h+2)\int_{\Sigma}\partial_{\bar{u}}G(u, x)\partial_w G(w, x) \\ & \times (Y_{IJ}^{-1}Y_{KL}^{-1} - Y_{IL}^{-1}Y_{JK}^{-1})\omega_I(x)\overline{\omega_J(x)}\omega_K(w)\overline{\omega_L(u)} \\ \psi_B = & \frac{\pi^2}{2}\int_{\Sigma^2}G(x, y)Y_{CD}^{-1}\omega_C(x)\overline{\omega_D(u)}Y_{AB}^{-1}\omega_A(w)\overline{\omega_B(y)} \\ & \times (Y_{IJ}^{-1}Y_{KL}^{-1} - Y_{IL}^{-1}Y_{JK}^{-1})\omega_I(w)\omega_K(y)\overline{\omega_J(x)}\overline{\omega_L(u)} \\ \psi_C = & \frac{2\pi}{4h}\int_{\Sigma^2}\partial_{\bar{u}}G(u, x)\partial_w G(w, y)P_{IJKL}Y_{AB}^{-1}\omega_I(x)\overline{\omega_L(y)} \\ & \times \left\{\omega_K(w)\omega_A(y) - \omega_K(y)\omega_A(w)\right\}\left\{\overline{\omega_J(u)}\overline{\omega_B(x)} - \overline{\omega_J(x)}\overline{\omega_B(u)}\right\} \end{aligned} \quad (4.11)$$

On general grounds, the mixed derivative  $\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi$  satisfies the following three conditions,

1. Hermiticity, namely invariance under  $w \leftrightarrow \bar{u}$ ;
2. Holomorphicity in  $w$ , namely  $\partial_{\bar{w}}(\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi) = 0$ ;
3. Holomorphicity in  $\bar{u}$ , namely  $\partial_u(\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi) = 0$ .

Hermiticity is seen to hold for each contribution  $\psi_A, \psi_B, \psi_C$  separately. Property 3 then follows from property 2, which in turn is proven in appendix B. Note that the holomorphicity properties are manifest for  $\psi_B$ , while in  $\psi_A$  and  $\psi_C$  they follow (at least in part) in view of the fact that the poles in the derivatives  $\partial_w G$  and  $\partial_{\bar{u}}G$  are cancelled by manifest zeros of corresponding combinations of Abelian differentials.

#### 4.5 Calculation of mixed variations for genus two

For genus two, we may use the special properties of  $h = 2$  to further simplify the expressions for  $\psi_A, \psi_B, \psi_C$ . We shall make use of the bi-form  $\Delta(x, y)$  which is defined by,

$$\omega_I(x) \omega_J(y) - \omega_J(x) \omega_I(y) = \varepsilon_{IJ} \Delta(x, y) \quad (4.12)$$

Here, we have  $I, J = 1, 2$ , and we use the convention  $\varepsilon_{12} = 1$ . The bi-form  $\Delta$  is a holomorphic section of the canonical bundle  $K$  in both  $x$  and  $y$ , and by construction its existence is limited to genus-two. Its zeros are at  $x = y$  and  $x = I(y)$  where  $I(y)$  is the image of  $y$  under the hyper-elliptic involution of the genus-two surface  $\Sigma$ . We shall also use the following relation, which is again special to genus-two,

$$Y_{IJ}^{-1} Y_{KL}^{-1} - Y_{IL}^{-1} Y_{JK}^{-1} = \varepsilon_{IK} \varepsilon_{JL} (\det Y)^{-1} \quad (4.13)$$

With the help of these relations we derive the following simplified expressions for  $h = 2$ ,

$$\begin{aligned} \psi_A &= -\frac{3\pi}{2} (\det Y)^{-1} \int_{\Sigma} \partial_{\bar{u}} G(u, x) \partial_w G(w, x) \Delta(x, w) \overline{\Delta(x, u)} \\ \psi_B &= -\frac{\pi^2}{2} (\det Y)^{-1} \int_{\Sigma^2} G(x, y) Y_{AB}^{-1} Y_{CD}^{-1} \omega_A(w) \overline{\omega_B(y)} \omega_C(x) \overline{\omega_D(u)} \Delta(y, w) \overline{\Delta(x, u)} \\ \psi_C &= +\frac{3\pi}{4} (\det Y)^{-1} \int_{\Sigma^2} \partial_{\bar{u}} G(u, x) \partial_w G(w, y) \Delta(x, w) \overline{\Delta(x, u)} Y_{IJ}^{-1} \omega_I(x) \overline{\omega_J(y)} \end{aligned} \quad (4.14)$$

The next key observation is that, using the first line of (3.3), we may combine the first and the last lines of (4.14) as follows,

$$\psi_A + \psi_C = -\frac{3}{4} (\det Y)^{-1} \int_{\Sigma^2} \partial_{\bar{u}} G(u, x) \partial_w G(w, y) \Delta(y, w) \overline{\Delta(x, u)} \partial_x \partial_{\bar{y}} G(x, y) \quad (4.15)$$

Upon integrating by parts in both  $x$  and  $\bar{y}$ , and using the holomorphicity of  $\Delta(y, w)$  in  $y$  and of  $\Delta(x, u)$  in  $x$ , we obtain an integral involving the product  $\partial_x \partial_{\bar{u}} G(u, x) \partial_{\bar{y}} \partial_w G(w, y)$ . Using again (3.3) on both mixed derivative factors, and exploiting the fact that the  $\delta$ -function contributions vanish, we find,

$$\psi_A + \psi_C = -\frac{3\pi^2}{4} \frac{Y_{AB}^{-1} Y_{CD}^{-1}}{\det Y} \int_{\Sigma^2} G(x, y) \omega_A(w) \overline{\omega_B(y)} \omega_C(x) \overline{\omega_D(u)} \Delta(y, w) \overline{\Delta(x, u)} \quad (4.16)$$

We recognize that this expression is proportional to  $\psi_B$ , so that our final formula becomes,

$$\delta_{\bar{u}\bar{u}} \delta_{ww} \varphi = -\frac{5\pi^2}{4} \frac{Y_{AB}^{-1} Y_{CD}^{-1}}{\det Y} \int_{\Sigma^2} G(x, y) \omega_A(w) \overline{\omega_B(y)} \omega_C(x) \overline{\omega_D(u)} \Delta(y, w) \overline{\Delta(x, u)} \quad (4.17)$$

In this form, hermiticity and holomorphicity in  $w$  and  $\bar{u}$  are manifest properties.

#### 4.6 Calculation of $\Delta\varphi$ for genus two

The form  $\delta_{\bar{u}\bar{u}} \delta_{ww} \varphi$  is a holomorphic quadratic differential in  $w$  and in  $\bar{u}$ . For genus-two, a basis of holomorphic quadratic differentials may be chosen in terms of the Abelian



differentials, namely  $\omega_I(w)\omega_J(w)$  in  $w$  for  $I \leq J$ , and similarly in  $\bar{u}$ . To exhibit this dependence systematically, we introduce the following notation,

$$\begin{aligned}\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi &= 4\pi^2\omega_I(w)\omega_J(w)\overline{\omega_K(u)}\overline{\omega_L(u)}T_{IJ;KL|AB;CD}\Phi_{AB;CD} \\ \Phi_{AB;CD} &= -\frac{5}{64}\int_{\Sigma^2}G(x,y)\omega_A(x)\overline{\omega_B(x)}\omega_C(y)\overline{\omega_D(y)}\end{aligned}\quad (4.18)$$

The tensor  $T$  is defined as follows for all genera,

$$\begin{aligned}T_{IJ;KL|AB;CD} &= +Y_{ID}^{-1}Y_{KA}^{-1}\left(Y_{JL}^{-1}Y_{BC}^{-1}-Y_{JB}^{-1}Y_{LC}^{-1}\right) \\ &\quad +Y_{JD}^{-1}Y_{KA}^{-1}\left(Y_{IL}^{-1}Y_{BC}^{-1}-Y_{IB}^{-1}Y_{LC}^{-1}\right) \\ &\quad +Y_{ID}^{-1}Y_{LA}^{-1}\left(Y_{JK}^{-1}Y_{BC}^{-1}-Y_{JB}^{-1}Y_{KC}^{-1}\right) \\ &\quad +Y_{JD}^{-1}Y_{LA}^{-1}\left(Y_{IK}^{-1}Y_{BC}^{-1}-Y_{IB}^{-1}Y_{KC}^{-1}\right)\end{aligned}\quad (4.19)$$

Note that the four terms in  $T$  arise from the symmetrization conditions in  $I, J$  and  $K, L$ .

From the above formula, and the variational formula for the period matrix in (4.2), we deduce the partial derivatives of  $\varphi$  with respect to  $\Omega$  and its complex conjugate,

$$\bar{\partial}_{KL}\partial_{IJ}\varphi = T_{IJ;KL|AB;CD}\Phi_{AB;CD}\quad (4.20)$$

The Laplacian, defined in (A.9), may now be applied to  $\varphi$ , and we find,

$$\Delta\varphi = 4Y_{IK}Y_{JL}T_{IJ;KL|AB;CD}\Phi_{AB;CD}\quad (4.21)$$

The contraction of the tensors yields the tensor  $P_{ABCD}$  defined in (3.1),

$$Y_{IK}Y_{JL}T_{IJ;KL|AB;CD} = 2P_{ABCD}\quad (4.22)$$

Putting all together, and using the definition and normalization of  $\varphi$  in (1.3), we derive the Laplace eigenvalue equation for  $\varphi$  that we had set out to prove,

$$\Delta\varphi = 5\varphi\quad (4.23)$$

We note that for genus higher than 2, no such simple expression appears to be available. In appendix C, we push the calculation of the corresponding equation for genus  $h \geq 3$  as far as possible. From a purely string theory point of view, of course, the ZK invariant is a natural object for genus-two, but probably not for higher genus.

## 5 Integrating the ZK invariant over moduli space

The purpose of this section is to provide a first principles calculation of the integral of the ZK invariant  $\varphi$  over the moduli space  $\mathcal{M}_2$  of genus-two compact Riemann surfaces,  $\int_{\mathcal{M}_2}d\mu_2\varphi$ , and thus to prove directly from superstring perturbation theory the value for this integral in (1.1) and (1.2) predicted from the interplay between S-duality and supersymmetry.

The key new ingredient we shall use here is the Laplace eigenvalue equation (4.23) satisfied by  $\varphi$  in the interior of  $\mathcal{M}_2$ . This equation allows us to recast the integral of  $\varphi$  over  $\mathcal{M}_2$  in terms of an integral of  $\Delta\varphi$  over  $\mathcal{M}_2$ , and this last integral can be reduced to an integral over the boundary  $\partial\mathcal{M}_2$  of moduli space. We now proceed to do so, properly taking into account convergence issues and contributions from the boundary  $\partial\mathcal{M}_2$ .

### 5.1 Convergence and regularization near the separating node

The integral  $\int_{\mathcal{M}_2} d\mu_2 \varphi$  is absolutely convergent, a property established already in [27]. Convergence may be verified explicitly by recalling the behavior of the volume form  $d\mu_2$  and the ZK invariant  $\varphi$  near the separating and non-separating components of the Deligne-Mumford compactification divisor of  $\mathcal{M}_2$ . To this end, it will be convenient to parametrize the period matrix  $\Omega$  and the volume form  $d\mu_2$  as follows,

$$\Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix} \quad d\mu_2 = \frac{d^2\tau \, d^2\tau_1 \, d^2\tau_2}{(\det Y)^3} \quad (5.1)$$

where  $d^2\tau = id\tau \wedge d\bar{\tau}$  and so on. To leading order, the asymptotics of the volume form  $d\mu_2$  is governed by the following expansions,

$$\begin{array}{ll} \text{separating} & \det Y = \text{Im}(\tau_1) \text{Im}(\tau_2) + \mathcal{O}(\tau^2) \\ \text{non-separating} & \det Y = \text{Im}(\tau_1) \text{Im}(\tau_2) + \mathcal{O}(\tau_2^0) \end{array} \quad (5.2)$$

The asymptotics of  $\varphi$  near the separating node is given by,

$$\varphi(\Omega) = -\ln \left| 2\pi\tau \, \eta(\tau_1)^2 \eta(\tau_2)^2 \right| + \mathcal{O}(|\tau|^2 \ln |\tau|) \quad (5.3)$$

while near the non-separating node  $\varphi$  has the following asymptotics, derived in [27] and [45] with the help of the degeneration results of [46],

$$\varphi(\Omega) = \frac{\pi}{6} (\text{Im} \tau_2) + \frac{5\pi}{6} \frac{(\text{Im} \tau)^2}{(\text{Im} \tau_1)} - \ln \left| \frac{\vartheta_1(\tau, \tau_1)}{\eta(\tau_1)} \right| + \mathcal{O}\left((\text{Im} \tau_2)^{-1}\right) \quad (5.4)$$

Using the parametrization in terms of  $\tau, \tau_1, \tau_2$  of the fundamental domain for  $\mathcal{M}_2$  given in appendix A, it follows by inspection that the integral  $\int_{\mathcal{M}_2} d\mu_2 \varphi$  is absolutely convergent.

To circumvent having to deal with modifications supported on the boundary  $\partial\mathcal{M}_2$  to the Laplace eigenvalue equation  $(\Delta - 5)\varphi = 0$ , we shall work with a regularized integral, which is kept away from the boundary. We shall prove below that no contributions arise from the non-separating node, so we need to regularize only near the separating node. To this end, we introduce the regularized domain for moduli space, defined by,

$$\mathcal{M}_2^\varepsilon = \mathcal{M}_2 \cap \left\{ \tau \in \mathbb{C}, \, |\tau| > \varepsilon \right\} \quad (5.5)$$

Everywhere on the space  $\mathcal{M}_2^\varepsilon$ , the function  $\varphi$  satisfies  $\Delta\varphi - 5\varphi = 0$ , just as it did in the interior of  $\mathcal{M}_2$ . Since the integral  $\int_{\mathcal{M}_2} d\mu_2 \varphi$  over all of moduli space is absolutely

convergent, we may recast it as a limit as  $\varepsilon \rightarrow 0$  of integrals over  $\mathcal{M}_2^\varepsilon$  instead, and for finite  $\varepsilon$  use the Laplace eigenvalue equation,

$$\int_{\mathcal{M}_2} d\mu_2 \varphi = \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}_2^\varepsilon} d\mu_2 \varphi = \frac{1}{5} \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}_2^\varepsilon} d\mu_2 \Delta \varphi \quad (5.6)$$

This equation will be the starting point for reducing the integral of the ZK invariant to an integral over the boundary of the regularized moduli space  $\mathcal{M}_2^\varepsilon$ .

## 5.2 Reducing the integral to the boundary of moduli space

To analyze the contribution from the boundary of moduli space arising from  $\Delta \varphi$ , we use the following form of the Laplacian acting on scalars,

$$(\det Y)^{-3} \Delta = 2\bar{\partial}_{IJ} \left( (\det Y)^{-3} Y_{IK} Y_{JL} \partial_{KL} \right) + \text{c.c.} \quad (5.7)$$

The formula follows directly from the usual differential geometry expression for the Laplacian with metric  $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$  being given by  $\sqrt{g} \Delta = \partial_\alpha (\sqrt{g} g^{\alpha\beta} \partial_\beta)$  where  $g = \det g_{\alpha\beta}$ . It may also be easily verified directly with the help of (A.9). It will be convenient to recast the formula using the following notations,

$$(\det Y)^{-3} \Delta = 2\partial_{\bar{\tau}_1} \mathcal{D}_{\tau_1} + 2\partial_{\bar{\tau}_2} \mathcal{D}_{\tau_2} + 2\partial_{\bar{\tau}} \mathcal{D}_\tau + \text{c.c.} \quad (5.8)$$

where the first order differential operators  $\mathcal{D}$  are defined by,

$$\begin{aligned} \mathcal{D}_{\tau_1} &= (\det Y)^{-3} Y_{1K} Y_{1L} \partial_{KL} \\ \mathcal{D}_{\tau_2} &= (\det Y)^{-3} Y_{2K} Y_{2L} \partial_{KL} \\ \mathcal{D}_\tau &= (\det Y)^{-3} Y_{1K} Y_{2L} \partial_{KL} \end{aligned} \quad (5.9)$$

Fortunately, we may answer the issue of boundary contributions by using only the leading asymptotic behaviour of  $\Delta$ . Sub-leading terms are typically difficult to compute. Keeping only the leading behaviour of the pre-factor  $\det Y$ , but exactly in all other contributions, we then have for both degenerations, and in terms of the coordinates  $\tau, \tau_1, \tau_2$ ,

$$\begin{aligned} \mathcal{D}_{\tau_1} &= (\text{Im } \tau_1)^{-3} (\text{Im } \tau_2)^{-3} \left( (\text{Im } \tau_1)^2 \partial_{\tau_1} + (\text{Im } \tau_1)(\text{Im } \tau) \partial_\tau + (\text{Im } \tau)^2 \partial_{\tau_2} \right) \\ \mathcal{D}_{\tau_2} &= (\text{Im } \tau_1)^{-3} (\text{Im } \tau_2)^{-3} \left( (\text{Im } \tau_2)^2 \partial_{\tau_2} + (\text{Im } \tau_2)(\text{Im } \tau) \partial_\tau + (\text{Im } \tau)^2 \partial_{\tau_1} \right) \\ \mathcal{D}_\tau &= (\text{Im } \tau_1)^{-3} (\text{Im } \tau_2)^{-3} \left( \frac{1}{2} \{ (\text{Im } \tau_1)(\text{Im } \tau_2) + (\text{Im } \tau)^2 \} \partial_\tau \right. \\ &\quad \left. + (\text{Im } \tau_1)(\text{Im } \tau) \partial_{\tau_1} + (\text{Im } \tau_2)(\text{Im } \tau) \partial_{\tau_2} \right) \end{aligned} \quad (5.10)$$

We must now investigate the behavior of  $\mathcal{D}_\tau \varphi$  as  $\tau \rightarrow 0$ , while keeping  $\tau_1, \tau_2$  fixed for the separating node, and the behavior of  $\mathcal{D}_{\tau_2} \varphi$  as  $\tau_2 \rightarrow i\infty$ , while keeping  $\tau, \tau_1$  fixed for the non-separating node. To do so, we use the asymptotic behaviors of (5.3) and (5.4).

Near the separating node, we find a pole as  $\tau \rightarrow 0$ ,

$$\mathcal{D}_\tau \varphi = (\text{Im } \tau_1)^{-2} (\text{Im } \tau_2)^{-2} \left( -\frac{1}{4\tau} + \mathcal{O}(|\tau| \ln |\tau|) \right) \quad (5.11)$$

Near the non-separating node, we find a contribution that tends to zero as  $\tau_2 \rightarrow i\infty$ ,

$$\mathcal{D}_{\tau_2}\varphi = -\frac{i\pi}{12}(\text{Im } \tau_1)^{-3}(\text{Im } \tau_2)^{-1} + \mathcal{O}\left((\text{Im } \tau_2)^{-2}\right) \quad (5.12)$$

We conclude from this that the non-separating degeneration node does not contribute to the integral of  $\Delta\varphi$ . On the other hand, however, there is a contribution from a pole at the separating degeneration node.

### 5.3 Calculation of the integral $\int d\mu_2 \varphi$

In the preceding sections, we have shown that the integral of (5.6) receives contributions only from the pole that arises at the separating node, while the contribution from the non-separating node vanishes identically. To extract the contribution from the pole at the separating node, we make use of the boundary expression for the Laplacian,

$$(\det Y)^{-3}\Delta\varphi \approx 2\partial_{\bar{\tau}}\mathcal{D}_{\tau}\varphi + 2\partial_{\tau}\mathcal{D}_{\bar{\tau}}\varphi \quad (5.13)$$

or in terms of the measure,

$$d\mu_2 \Delta\varphi \approx 2d^2\tau d^2\tau_1 d^2\tau_2 (\partial_{\bar{\tau}}\mathcal{D}_{\tau}\varphi + \partial_{\tau}\mathcal{D}_{\bar{\tau}}\varphi) \quad (5.14)$$

where we use the convention  $d^2\tau = i d\tau \wedge d\bar{\tau}$ . By integration by parts in  $\tau$  over  $\mathcal{M}_2^\varepsilon$ , the boundary contribution localizes at  $|\tau| = \varepsilon$ . To extract it, we make use of the relation,

$$i d\tau \wedge d\bar{\tau} (\partial_{\bar{\tau}}\mathcal{D}_{\tau}\varphi + \partial_{\tau}\mathcal{D}_{\bar{\tau}}\varphi) = d(-i d\tau \mathcal{D}_{\tau}\varphi + i d\bar{\tau} \mathcal{D}_{\bar{\tau}}\varphi) \quad (5.15)$$

so that

$$\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{2}{5} \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{M}_2^\varepsilon} d^2\tau_1 d^2\tau_2 \left( i d\tau \mathcal{D}_{\tau}\varphi - i d\bar{\tau} \mathcal{D}_{\bar{\tau}}\varphi \right) \quad (5.16)$$

where  $\partial\mathcal{M}_2^\varepsilon$  stands for the boundary  $|\tau| = \varepsilon$  near the separating node, and is given by,

$$\partial\mathcal{M}_2^\varepsilon = \{\tau \in \mathbb{C}, |\tau| = \varepsilon\} \times (\mathcal{M}_1 \times \mathcal{M}_1)/(\mathbb{Z}_2 \times \mathbb{Z}_2) \quad (5.17)$$

Using the above results for  $\mathcal{D}_{\tau}\varphi$ ,

$$\mathcal{D}_{\tau}\varphi = (\text{Im } \tau_1)^{-2}(\text{Im } \tau_2)^{-2}v_{\tau} \quad v_{\tau} = -\frac{1}{4\tau} \quad (5.18)$$

and the value for the genus-one volume,

$$\int_{\mathcal{M}_1} \frac{|d\tau_i|^2}{(\text{Im } \tau_i)^2} = \frac{2\pi}{3} \quad (5.19)$$

for  $i = 1, 2$ , we may perform the integrations over  $\tau_1$  and  $\tau_2$  first, and we get,

$$\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{2}{5} \times \frac{1}{2^2} \times \left(\frac{2\pi}{3}\right)^2 \lim_{\varepsilon \rightarrow 0} \oint_{|\tau|=\varepsilon} \left( i d\tau v_{\tau} - i d\bar{\tau} v_{\bar{\tau}} \right) \quad (5.20)$$

The value of the integral on the r.h.s. is simply  $\pi$ , so that we have,

$$\int_{\mathcal{M}_2} d\mu_2 \varphi = \frac{2\pi^3}{45} \quad (5.21)$$

Upon multiplying both sides by a factor of  $\pi$ , we reproduce the value announced in (1.1) and predicted in [27].

#### 5.4 Differential relation for the Faltings invariant

In this subsection, we shall provide immediate mathematical implications of the relations derived above. First we extend the validity of the eigenvalue equation for  $\varphi$  to the compactified moduli space  $\overline{\mathcal{M}}_2$ . Second, we derive Laplace eigenvalue equations for two other Siegel modular forms, including the Faltings invariant for genus-two surfaces.

To begin, we complete the analysis of the ZK invariant by quoting the Laplace eigenvalue equation valid on the Deligne-Mumford compactification  $\overline{\mathcal{M}}_2$  of moduli space. Combining the result  $(\Delta - 5)\varphi = 0$  of (4.23) with the observation that the contribution from the non-separating node vanishes, while the contribution of the separating node results from combining equations (5.8), and (5.11), we find,

$$\Delta\varphi - 5\varphi = -2\pi\delta_{SN} \quad \delta_{SN} = (\det Y) \delta^{(2)}(\tau) \quad (5.22)$$

Here the Dirac  $\delta$ -function has been normalized to  $\int d^2\tau \delta^{(2)}(\tau) = 1$ , and  $\delta_{SN}$  is the induced  $\delta$ -function on the separating node.

The relation of  $\varphi(\Omega)$  with the Faltings invariant  $\delta(\Omega)$  was established in [30],

$$\varphi(\Omega) = 36 \ln 2 - 40 \ln(2\pi) - 3 \ln \|\Psi_{10}(\Omega)\| - \frac{5}{2} \delta(\Omega) \quad (5.23)$$

Here,  $\Psi_{10}$  is the genus-two cusp form of Igusa, and its Peterson norm is defined by,

$$\|\Psi_{10}(\Omega)\| = (\det Y)^5 |\Psi_{10}(\Omega)| \quad (5.24)$$

The Laplacian of its logarithm is given by,

$$\Delta \ln \|\Psi_{10}(\Omega)\| = -15 + 4\pi\delta_{SN} \quad (5.25)$$

The  $\delta_{SN}$ -function comes from the separating node, where  $\ln |\Psi_{10}(\Omega)| \approx \ln |\tau|^2$ . An alternative expression for  $\varphi$  was obtained in [27] in terms of an integral over the Jacobian of  $\Sigma$ ,

$$\begin{aligned} \varphi(\Omega) &= 6 \ln 2 - \frac{1}{4} \ln |\Psi_{10}(\Omega)|^2 + 5 \ln \Phi(\Omega) \\ \ln \Phi(\Omega) &= \int_{\mathbb{T}^4} d^4x \ln \left| \vartheta[x](0, \Omega) \right|^2 \end{aligned} \quad (5.26)$$

where the characteristics  $x$  take values in the unit square torus  $\mathbb{T}^4$ .

It is now immediate to obtain the corresponding relations for the Faltings invariant and for the modular form  $\Phi$ , and we have,

$$\begin{aligned} (\Delta - 5) \delta(\Omega) &= \varphi_1 + 6 \ln \|\Psi_{10}(\Omega)\| - 4\pi\delta_{SN} \\ (\Delta - 5) \ln \Phi(\Omega) &= 6 \ln 2 - \frac{1}{2} \ln |\Psi_{10}(\Omega)| \end{aligned} \quad (5.27)$$

where  $\varphi_1 = 18 - 72 \ln 2 + 80 \ln(2\pi)$ . Note that the equation for  $\Phi$  has vanishing contribution from the separating divisor, as is indeed consistent with its regularity at this node [27].

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## A Some modular geometry

The Siegel upper half space  $\mathcal{S}_h$  of rank  $h$  is defined by,

$$\mathcal{S}_h = \{\Omega_{IJ} = \Omega_{JI}, \ 1 \leq I, J \leq h, \quad \text{Im} \, \Omega > 0\} \quad (\text{A.1})$$

Symplectic transformations  $M \in \text{Sp}(2h, \mathbb{R})$  are defined by the relations,

$$M^t J M = J \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad J = \begin{pmatrix} 0 & I_h \\ -I_h & 0 \end{pmatrix} \quad (\text{A.2})$$

The action of  $\text{Sp}(2h, \mathbb{R})$  on  $\Omega$  is given by,

$$\Omega \rightarrow M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1} \quad (\text{A.3})$$

The isotropy group of any point in  $\mathcal{S}_h$  is isomorphic to  $\text{U}(h)$ , so that  $\mathcal{S}_h$  may also be viewed as a coset  $\mathcal{S}_h = \text{Sp}(2h, \mathbb{R})/\text{U}(h)$ . We decompose  $\Omega$  into real matrices  $X, Y$  via  $\Omega = X + iY$ , and to use the abbreviation  $Y_{IJ}^{-1} = (Y^{-1})_{IJ}$ .

### A.1 Metric and volume

The Poincaré metric on  $\mathcal{S}_h$  is constructed to be invariant under  $\text{Sp}(2h, \mathbb{R})$ , and is given by

$$ds_h^2 = \sum_{I,J,K,L=1}^h Y_{IJ}^{-1} d\bar{\Omega}_{JK} Y_{KL}^{-1} d\Omega_{LI} \quad (\text{A.4})$$

The associated invariant volume form  $d\mu_h$  is defined by,

$$d\mu_h = \frac{1}{(\det Y)^{h+1}} \bigwedge_{I \leq J} i d\Omega_{IJ} \wedge d\bar{\Omega}_{IJ} \quad (\text{A.5})$$

It may be readily verified, for example, that  $ds_h^2$  and  $d\mu_h$  are invariant under the scaling transformation  $\Omega \rightarrow \lambda^2 \Omega$ , for which we have  $A = D^{-1} = \lambda I_h$  and  $B = C = 0$ .

The Siegel fundamental domain  $\mathcal{F}_h$  is the quotient of the Siegel upper half space  $\mathcal{S}_h$  by the modular group  $\mathrm{Sp}(2h, \mathbb{Z})/\mathbb{Z}_2$ . Its volume  $V_h = \int_{\mathcal{F}_h} d\mu_h$  was calculated by Siegel [47], and is given by,<sup>10</sup>

$$V_h = 2 \prod_{k=1}^h \left( \frac{2^k}{\pi^k} \Gamma(k) \zeta(2k) \right) \quad (\text{A.6})$$

where  $\zeta(2k)$  is the Riemann zeta function. Its lowest values for even argument are given by,

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(4) = \frac{\pi^4}{90} \quad \zeta(6) = \frac{\pi^6}{945} \quad (\text{A.7})$$

which results in the following values of the volumes,

$$V_1 = \frac{2\pi}{3} \quad V_2 = \frac{4\pi^3}{3^3 5} \quad V_3 = \frac{2^6 \pi^6}{3^6 5^2 7} \quad (\text{A.8})$$

## A.2 The Laplace-Beltrami operator

The  $\mathrm{Sp}(2h, \mathbb{R})$ -invariant Laplacian on  $\mathcal{S}_h$ , which is associated with the Poincaré metric, was derived in [48] and is given by,

$$\Delta_{\mathrm{Sp}(2h)} = \sum_{I,J,K,L=1}^h 4 Y_{IK} Y_{JL} \bar{\partial}_{KL} \partial_{IJ} \quad (\text{A.9})$$

Throughout, we shall use the standard composite index notation  $\partial_{IJ}$  for the partial derivatives with respect to  $\Omega$ , defined by,

$$\partial_{IJ} \equiv \frac{1}{2} (1 + \delta_{IJ}) \frac{\partial}{\partial \Omega_{IJ}} \quad (\text{A.10})$$

along with their complex conjugates  $\bar{\partial}_{IJ}$ . The above definition of the derivative in  $\Omega$  guarantees that the derivative behaves in a tensorial manner under modular transformations, so that for example we have

$$\partial_{IJ} \Omega_{KL} = \frac{1}{2} (\delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK}) \quad (\text{A.11})$$

We note that the Laplacian is normalised so that,

$$\Delta_{\mathrm{Sp}(2h)} (\det Y)^s = \frac{1}{2} h s (2s - h - 1) (\det Y)^s \quad (\text{A.12})$$

It readily follows that  $\Delta_{\mathrm{Sp}(2h)} \ln(\det Y) = -\frac{1}{2} h (h + 1)$ .

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<sup>10</sup>The normalization of the volume form in (A.5) includes an extra factor of 2 for each complex moduli dimension, as compared to the volume form used in [47] and [35, 37].



### A.3 Moduli spaces of low genus

We give a synopsis of the moduli spaces  $\mathcal{M}_h$  of compact Riemann surfaces  $\Sigma$  at low genus  $h$  and their representation as fundamental domains  $\mathcal{F}_h$  in  $\mathcal{S}_h$ . This will allow us to easily identify their canonical metric, volume form, and volume. To define these ingredients, we fix a homology basis of 1-cycles  $A_I, B_I$  with  $I = 1, \dots, h$  in  $H_1(\Sigma, \mathbb{Z})$ , with canonical intersection pairing  $\#(A_I, A_J) = \#(B_I, B_J) = 0$ ,  $\#(A_I, B_J) = \delta_{IJ}$ , and a dual basis of holomorphic 1-forms  $\omega_I$  in  $H^1(\Sigma, \mathbb{C})$  subject to the canonical normalization on  $A_I$ -cycles,

$$\oint_{A_I} \omega_J = \delta_{IJ} \qquad \oint_{B_I} \omega_J = \Omega_{IJ} \qquad (\text{A.13})$$

The period matrix  $\Omega$  belongs to  $\mathcal{S}_h$ . For a given Riemann surface, the period matrix is defined up to a modular transformation  $M \in \text{Sp}(2h, \mathbb{Z}) \subset \text{Sp}(2h, \mathbb{R})$ , which corresponds to a redefinition of the canonical homology basis in  $H_1(\Sigma, \mathbb{Z})$ , with integer coefficients. Note that the element  $-I \in \text{Sp}(2h, \mathbb{Z})$  leaves every  $\Omega$  invariant, so the identification is more properly under  $\text{Sp}(2h, \mathbb{Z})/\mathbb{Z}_2$ .

The moduli space  $\mathcal{M}_1$  of genus-one compact Riemann surfaces coincides with the fundamental domain of  $\text{Sp}(2, \mathbb{Z})/\mathbb{Z}_2$  in  $\mathcal{S}_1$  and is given explicitly by,

$$\mathcal{M}_1 = \mathcal{F}_1 = \left\{ \tau \in \mathbb{C}; \text{Im}(\tau) > 0, |\tau| \geq 1, |\text{Re}(\tau)| \leq \frac{1}{2} \right\} \qquad (\text{A.14})$$

The  $\text{Sp}(2, \mathbb{R})$ -invariant Poincaré metric  $ds_1^2$  is that of (A.4) for genus-one, the associated volume form  $d\mu_1$  is that of (A.5) for genus one, and total volume is  $V_1$ , as given in (A.8).

The moduli space  $\mathcal{M}_2$  for compact genus-two Riemann surfaces may be identified with the fundamental domain  $\mathcal{F}_2$  of  $\text{Sp}(4, \mathbb{Z})/\mathbb{Z}_2$  in  $\mathcal{S}_2$ . Actually, the separating node  $\mathcal{M}_1 \times \mathcal{M}_1$  must be removed, as it does not correspond to a compact surface.  $\mathcal{M}_2$  may be described concretely by the following set of inequalities on  $\Omega = X + iY$ , which were established in [49] and were reviewed in [50],

$$\begin{aligned} (1) \quad & -\frac{1}{2} \leq X_{11}, X_{12}, X_{22} \leq +\frac{1}{2} \\ (2) \quad & 0 < 2Y_{12} \leq Y_{11} \leq Y_{22} \\ (3) \quad & |\det(C\Omega + D)| \geq 1 \text{ for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}) \end{aligned} \qquad (\text{A.15})$$

The  $\text{Sp}(4, \mathbb{R})$ -invariant Poincaré metric  $ds_2^2$  is that of (A.4) for genus-two, and the associated volume form  $d\mu_2$  is that of (A.8) for genus-two. The volume  $V_2$  is given in (A.8).

The moduli space  $\mathcal{M}_3$  for compact genus-three Riemann surfaces may be identified with the fundamental domain of  $\text{Sp}(6, \mathbb{Z})/\mathbb{Z}_2$  in  $\mathcal{S}_3$ . Actually, this identification is two-to-one, and requires the removal of the some sub-varieties. As a result, the volume is  $V_3/2$  where  $V_3$  was given in (A.8). For higher genus,  $h \geq 4$ , the dimensions of  $\mathcal{M}_h$  and  $\mathcal{S}_h$  are respectively  $3h - 3$  and  $h(h + 1)/2$  and no longer match. Instead,  $\mathcal{M}_h$  is then a complex sub-variety of  $\mathcal{S}_h$  specified by the Schottky relations.

## B Calculation of the mixed variation of $\varphi$

In this appendix, we provide the details of the calculation of the mixed variational derivatives  $\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi$ , and prove that this form is holomorphic in  $w$  and  $\bar{u}$ . The starting point is the expression for the first variational derivative of  $\varphi$  given in (4.5), along with the variational derivatives given in (4.8) and (4.9).

The contributions to  $\partial_{\bar{u}\bar{u}}\delta_{ww}\varphi$  may be split into those arising from the variations of the terms of the form  $P_{IJKL}\omega_I(s)\overline{\omega_J(x)}\omega_K(t)\overline{\omega_L(y)}$  in (4.5), and those arising from the variation of the derivatives of  $G$  in (4.5). In an obvious notations, we have,

$$\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi = \delta_{\bar{u}\bar{u}}\delta_{ww}\varphi\Big|_P + \delta_{\bar{u}\bar{u}}\delta_{ww}\varphi\Big|_G \quad (\text{B.1})$$

These contributions may be written explicitly. For the part involving  $P$ , we have,

$$\begin{aligned} \delta_{\bar{u}\bar{u}}\delta_{ww}\varphi\Big|_P = \frac{1}{2h} \int_{\Sigma^2} P_{IJKL} \Big\{ & \\ & - \omega_I(x) \overline{\omega_J(u)} \omega_K(y) \overline{\omega_L(y)} \partial_w G(w, x) \partial_w G(w, y) \partial_{\bar{u}} \partial_{\bar{x}} G(u, x) \\ & + \omega_I(w) \overline{\omega_J(u)} \omega_K(y) \overline{\omega_L(y)} \partial_w G(w, x) \partial_x G(x, y) \partial_{\bar{u}} \partial_{\bar{x}} G(u, x) \\ & + \omega_I(x) \overline{\omega_J(u)} \omega_K(w) \overline{\omega_L(y)} \partial_w G(w, y) \partial_y G(x, y) \partial_{\bar{u}} \partial_{\bar{x}} G(u, x) \Big\} \end{aligned} \quad (\text{B.2})$$

Here we have used the symmetry under the interchange between the integrations over  $x$  and  $y$ , and the indices  $P_{IJKL} = P_{KLIJ}$  to pairwise combine terms and bring out an overall factor of 2. For the part involving the variations of the derivatives of  $G$  in (4.5), one first establishes that the contributions from the terms in  $\bar{\partial}\delta$  in (4.8) vanish identically. The remaining part simplifies to give the following result,

$$\begin{aligned} \delta_{\bar{u}\bar{u}}\delta_{ww}\varphi\Big|_G = \frac{\pi}{2h} \int_{\Sigma^2} P_{IJKL} Y_{AB}^{-1} \Big\{ & \\ & + \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)} \omega_A(w) \overline{\omega_B(u)} \partial_{\bar{u}} G(u, x) \partial_w G(w, y) \\ & + \omega_I(w) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)} \omega_A(w) \overline{\omega_B(u)} G(x, y) \partial_{\bar{u}} \partial_x G(u, x) \\ & - \omega_I(w) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)} \omega_A(x) \overline{\omega_B(u)} \partial_{\bar{u}} G(u, y) \partial_w G(w, x) \Big\} \end{aligned} \quad (\text{B.3})$$

To obtain the cancellation of the  $\bar{\partial}\delta$  terms, and some further simplifications, we have used the fact that  $G$  is single-valued, so that integrations by parts can be performed without producing boundary terms (since  $\Sigma$  has no boundary), as well as the orthogonality relations of (3.5). Further simplifications are obtained as follows. In (B.2), we integrate by parts in  $\bar{x}$  to eliminate one of the Green function contributions using (3.3). The mixed derivatives that arise in the process produce  $\delta$ -functions which, using (B.7) below, yield  $\psi_A$ . They also produce terms that are of the same form as the terms in (B.3), and which are combined in the terms  $\psi_B$  and  $\psi_C$  below, as follows,

$$\delta_{\bar{u}\bar{u}}\delta_{ww}\varphi = \psi_A + \psi_B + \psi_C \quad (\text{B.4})$$

The expressions for  $\psi_A$  and  $\psi_C$  are those given in the main body of the paper (4.11). For  $\psi_B$ , we find,

$$\begin{aligned} \psi_B = \frac{2\pi}{4h} \int_{\Sigma^2} G(x, y) \partial_x \partial_{\bar{u}} G(u, x) P_{IJKL} Y_{AB}^{-1} \omega_I(w) \overline{\omega_L(y)} \\ \times \omega_K(y) \omega_A(w) \left\{ \overline{\omega_J(x) \omega_B(u)} - \overline{\omega_J(u) \omega_B(x)} \right\} \end{aligned} \quad (\text{B.5})$$

The single poles in the derivatives of the Green functions at  $x = u, w$  in  $\psi_A$  and at  $\bar{x} = \bar{u}$  and  $y = w$  in  $\psi_C$  are cancelled by the zeros of the Abelian differential factors due to the antisymmetry under the interchange of  $I, K$  and independently of  $J, L$ . The  $\delta(x, u)$ -function arising from the mixed derivative of  $G$  in  $\psi_B$  is similarly cancelled by the effect of antisymmetry in  $J, B$ . The last simplification leads to the expression for  $\psi_B$  in (4.11).

The mixed derivative  $\delta_{\bar{u}\bar{u}} \delta_{ww} \varphi$  must satisfy hermiticity, namely invariance under  $w \leftrightarrow \bar{u}$ , and holomorphicity in  $w$  and  $\bar{u}$ . Hermiticity is established by inspection of each contribution  $\psi_A, \psi_B, \psi_C$  separately. Holomorphicity in  $w$  is manifest for the contribution  $\psi_B$  in (4.11), as its only  $w$ -dependence is through the holomorphic Abelian differentials  $\omega_A(w) \omega_I(w)$ . The other contributions are readily evaluated using the second equation in (3.3),

$$\begin{aligned} \partial_{\bar{w}} \psi_A &= -\frac{2\pi^2}{4h} (2h+2) Y_{CD}^{-1} \omega_C(w) \overline{\omega_D(w)} \int_{\Sigma} \partial_{\bar{u}} G(u, x) \\ &\quad \times (Y_{IJ}^{-1} Y_{KL}^{-1} - Y_{IL}^{-1} Y_{JK}^{-1}) \omega_I(x) \overline{\omega_J(x)} \omega_K(w) \overline{\omega_L(u)} \\ \partial_{\bar{w}} \psi_C &= \frac{\pi^2}{2h} Y_{CD}^{-1} \omega_C(w) \overline{\omega_D(w)} \int_{\Sigma^2} \partial_{\bar{u}} G(u, x) P_{IJKL} Y_{AB}^{-1} \omega_I(x) \overline{\omega_L(y)} \\ &\quad \times \left\{ \omega_K(w) \omega_A(y) - \omega_K(y) \omega_A(w) \right\} \left\{ \overline{\omega_J(x) \omega_B(u)} - \overline{\omega_J(u) \omega_B(x)} \right\} \end{aligned} \quad (\text{B.6})$$

Carrying out the integration over  $y$  in  $\partial_{\bar{w}} \psi_C$ , and using (3.5) and the identity

$$P_{IJKB} - P_{IBKJ} = -(h+1) (Y_{IJ}^{-1} Y_{KB}^{-1} - Y_{IB}^{-1} Y_{JK}^{-1}) \quad (\text{B.7})$$

it is immediate that  $\partial_{\bar{w}} \psi_C = -\partial_{\bar{w}} \psi_A$ , thereby proving holomorphicity in  $w$ .

## C Calculation of the Laplacian of $\varphi$ for genus $h \geq 3$

Mathematically, the ZK invariant may be defined in terms of the scalar Green function for any  $h \geq 3$ . Physically, however, there is no compelling evidence at this time that the ZK invariant plays any role in superstring perturbation theory at genus  $h \geq 3$ . Even if it did, it is not even clear to which order  $p$  in  $D^{2p} \mathcal{R}^4$  it would contribute. Still, from our variational approach, we have access to evaluating the Laplacian on moduli space  $\mathcal{M}_h$  of  $\varphi$ , and we shall carry as far as possible its calculation in this appendix.

To evaluate  $\Delta \varphi$  for all genera, we shall derive a formula which isolates the part that contributes for  $h \geq 3$ , but vanishes identically for  $h = 2$ . It is in this form that the higher genus expression for  $\Delta \varphi$  will remain as close as possible to the genus-two formula. To do

so, we transform  $\psi_A$  in (4.14) into a double integral over  $\Sigma^2$ , just as  $\psi_B, \psi_C$  are double integrals, by using the second identity in (3.3). The resulting two contributions of  $\psi_A$  will be denoted respectively by  $\psi_A^1, \psi_A^2$  with,

$$\psi_A = \psi_A^1 + \psi_A^2 \quad (\text{C.1})$$

The double integrals are given as follows,

$$\begin{aligned} \psi_A^1 &= -\frac{2h+2}{4h} \int_{\Sigma^2} \partial_{\bar{u}} G(u, x) \partial_w G(w, x) \partial_x \partial_{\bar{y}} G(x, y) \\ &\quad \times (Y_{IJ}^{-1} Y_{KL}^{-1} - Y_{IL}^{-1} Y_{JK}^{-1}) \omega_I(y) \overline{\omega_J(x)} \omega_K(w) \overline{\omega_L(u)} \\ \psi_A^2 &= -\frac{\pi}{4h} (2h+2) \int_{\Sigma^2} \partial_{\bar{u}} G(u, x) \partial_w G(w, x) Y_{AB}^{-1} \omega_A(x) \overline{\omega_B(y)} \\ &\quad \times (Y_{IJ}^{-1} Y_{KL}^{-1} - Y_{IL}^{-1} Y_{JK}^{-1}) \omega_I(y) \overline{\omega_J(x)} \omega_K(w) \overline{\omega_L(u)} \end{aligned} \quad (\text{C.2})$$

Integrating by parts in  $x$  and  $\bar{y}$  in  $\psi_A^1$ , we recover a term proportional to  $\psi_B$ . With the help of the modular tensor  $T$ , originally defined in (4.19) for genus-two but now extended to all genera, and after some further simplifications, we obtain,

$$\begin{aligned} \psi_A^1 + \psi_B &= -\pi^2 \frac{2h+1}{8h} \omega_I(w) \omega_J(w) \overline{\omega_K(u)} \overline{\omega_L(u)} T_{IJKL|ABCD} \\ &\quad \times \int_{\Sigma^2} G(x, y) \omega_A(x) \overline{\omega_B(x)} \omega_C(y) \overline{\omega_D(y)} \end{aligned} \quad (\text{C.3})$$

We extract the part of the Laplacian due to  $\psi_A^1 + \psi_B$ , in a manner which generalizes the calculation of section 4.6 to arbitrary genus. In particular, we make use of the contraction formula (4.22) which was established for genus-two, but in fact holds unmodified for arbitrary genus. The result is as follows,

$$\Delta \varphi \Big|_{\psi_A^1 + \psi_B} = (2h+1) \varphi \quad (\text{C.4})$$

Note that this expression already saturates the equation for  $h=2$ , so we should expect the contribution from  $\psi_A^2 + \psi_C$  to the Laplacian to vanish for  $h=2$ .

### C.1 The contribution to $\Delta \varphi$ which vanishes for $h=2$

The remaining part  $\psi = \psi_A^2 + \psi_C$  of  $\delta_{\bar{u}\bar{u}} \delta_{ww} \varphi$  may be combined as follows,

$$\psi = \frac{2\pi}{h} S_{IJK;ABC} \int_{\Sigma^2} \partial_{\bar{u}} G(u, x) \partial_w G(w, y) \omega_I(x) \omega_{[J}(w) \omega_{K]}(y) \overline{\omega_{[A}(u)} \overline{\omega_{B]}(x)} \overline{\omega_C(y)} \quad (\text{C.5})$$

where the brackets  $[ ]$  instruct to anti-symmetrize the enclosed indices, namely  $JK$  in the first brackets, and  $AB$  in the second. The modular tensor  $S$  arising from combining equations (4.14) for  $\psi_C$  and (C.2) for  $\psi_A^2$  is given by the anti-symmetrization in the indices  $AB$  and  $JK$  of the expression,

$$-4Y_{IA}^{-1} Y_{JC}^{-1} Y_{KB}^{-1} - 2(h-1)Y_{IC}^{-1} Y_{JB}^{-1} Y_{KA}^{-1} \quad (\text{C.6})$$

Remarkably, the properly anti-symmetrized form may be simply expressed in terms of the unique rank-six, degree three anti-symmetric tensor of  $Y^{-1}$ , defined by,

$$\mathfrak{A}_{IJK;ABC} \equiv +Y_{IA}^{-1} Y_{JB}^{-1} Y_{KC}^{-1} + Y_{IB}^{-1} Y_{JC}^{-1} Y_{KA}^{-1} + Y_{IC}^{-1} Y_{JA}^{-1} Y_{KB}^{-1} \\ - Y_{IA}^{-1} Y_{JC}^{-1} Y_{KB}^{-1} - Y_{IB}^{-1} Y_{JA}^{-1} Y_{KC}^{-1} - Y_{IC}^{-1} Y_{JB}^{-1} Y_{KA}^{-1} \quad (\text{C.7})$$

One finds,

$$S_{IJK;ABC} = \mathfrak{A}_{IJK;ABC} + Y_{IC}^{-1} Y^{LD} \mathfrak{A}_{JKL;ABD} \quad (\text{C.8})$$

For  $h = 2$ , the tensor  $\mathfrak{A}_{IJK;ABC}$  vanishes identically, so that the contribution the  $\Delta\varphi$  originating from  $\psi$  vanishes identically as well. To check that  $\psi$  is holomorphic in  $w$  and  $\bar{u}$ , it suffices to make use of the identity,

$$Y^{IB} S_{IJK;ABC} = 0 \quad (\text{C.9})$$

and its permutations. The part of  $\Delta\varphi$  coming from  $\psi$  may be evaluated by using the holomorphicity of  $\psi$  in  $w$  and  $\bar{u}$  to write  $\psi$  as follows,

$$\psi = 4\pi^2 \omega_I(w) \omega_J(w) \overline{\omega_K(u)} \overline{\omega_L(u)} \Lambda_{IJ;KL} \quad (\text{C.10})$$

so that we obtain our final formula,

$$\Delta\varphi - (2h+1)\varphi = 4Y_{IK} Y_{JL} \Lambda_{IJ;KL} \quad (\text{C.11})$$

To extract  $\Lambda_{IJ;KL}$  from  $\psi$  is cumbersome, but straightforward. Using the fact that  $\psi$  is a holomorphic quadratic form in both  $w$  and  $\bar{u}$ , and that the set  $\omega_I(w)\omega_J(w)$  and  $\overline{\omega_K(u)}\overline{\omega_L(u)}$  for  $I, J, K, L = 1, \dots, h$  spans a basis for such forms, we may obtain the  $w$ -dependence by choosing  $3h-3$  generic points  $p_a$  at which to evaluate  $w$ , and  $3h-3$  generic points  $q_b$  at which to evaluate  $u$ .  $\Lambda_{IJ;KL}$  is then obtained by inverting the  $(3h-3) \times (3h-3)$ -dimensional matrix. The procedure will provide a unique modular tensor  $\Lambda_{IJ;KL}$  for  $h = 3$ , but will fail to give a unique result for  $h \geq 4$  due to the Schottky relations, which may be viewed as imposing  $\frac{1}{2}(h-2)(h-3)$  linear relations between the quadratic differentials  $\omega_I(w)\omega_J(w)$ .

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